For $n = 1$,

$$2 = c_1 \sinh \frac{\pi a}{b} \quad \text{or} \quad c_1 = \frac{2}{\sinh \frac{\pi a}{b}}$$

For $n = 5$,

$$\frac{1}{10} = c_5 \sinh \frac{5\pi a}{b} \quad \text{or} \quad c_5 = \frac{1}{10 \sinh \frac{5\pi a}{b}}$$

Hence,

$$V(x, y) = \frac{2 \sin \frac{\pi x}{b} \sinh \frac{\pi y}{b}}{\sinh \frac{\pi a}{b}} + \frac{\sin \frac{5\pi x}{b} \sinh \frac{5\pi y}{b}}{10 \sinh \frac{5\pi a}{b}}$$

PRACTICE EXERCISE 6.6

In Example 6.5, suppose everything remains the same except that V_o is replaced by $V_o \sin \frac{7\pi x}{b}$, $0 \leq x \leq b$, $y = a$. Find $V(x, y)$.

Answer: $\dfrac{V_o \sin \frac{7\pi x}{b} \sinh \frac{7\pi y}{b}}{\sinh \frac{7\pi a}{b}}$

EXAMPLE 6.7

Obtain the separated differential equations for potential distribution $V(\rho, \phi, z)$ in a charge-free region.

Solution:

This example, like Example 6.5, further illustrates the method of separation of variables. Since the region is free of charge, we need to solve Laplace's equation in cylindrical coordinates; that is,

$$\nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{6.7.1}$$

We let

$$V(\rho, \phi, z) = R(\rho)\, \Phi(\phi)\, Z(z) \tag{6.7.2}$$

where R, Φ, and Z are, respectively, functions of ρ, ϕ, and z. Substituting eq. (6.7.2) into eq. (6.7.1) gives

$$\frac{\Phi Z}{\rho}\frac{d}{d\rho}\left(\frac{\rho\, dR}{d\rho}\right) + \frac{RZ}{\rho^2}\frac{d^2\Phi}{d\phi^2} + R\Phi\frac{d^2Z}{dz^2} = 0 \tag{6.7.3}$$

We divide through by $R\Phi Z$ to obtain

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\frac{\rho\, dR}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} \tag{6.7.4}$$

The right-hand side of this equation is solely a function of z whereas the left-hand side does not depend on z. For the two sides to be equal, they must be constant; that is,

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\frac{\rho\, dR}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} = -\lambda^2 \tag{6.7.5}$$

where $-\lambda^2$ is a separation constant. Equation (6.7.5) can be separated into two parts:

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = \lambda^2 \tag{6.7.6}$$

or

$$Z'' - \lambda^2 Z = 0 \tag{6.7.7}$$

and

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\frac{\rho\, dR}{d\rho}\right) + \lambda^2\rho^2 + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0 \tag{6.7.8}$$

Equation (6.7.8) can be written as

$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} + \lambda^2\rho^2 = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \mu^2 \tag{6.7.9}$$

where μ^2 is another separation constant. Equation (6.7.9) is separated as

$$\Phi'' = \mu^2\Phi = 0 \tag{6.7.10}$$

and

$$\rho^2R'' + \rho R' + (\rho^2\lambda^2 - \mu^2)R = 0 \tag{6.7.11}$$

Equations (6.7.7), (6.7.10), and (6.7.11) are the required separated differential equations. Equation (6.7.7) has a solution similar to the solution obtained in Case 2 of Example 6.5; that is,

$$Z(z) = c_1 \cosh \lambda z + c_2 \sinh \lambda z \tag{6.7.12}$$

The solution to eq. (6.7.10) is similar to the solution obtained in Case 3 of Example 6.5; that is,

$$\Phi(\phi) = c_3 \cos \mu\phi + c_4 \sin \mu\phi \tag{6.7.13}$$

Equation (6.7.11) is known as the *Bessel differential equation* and its solution is beyond the scope of this text.[3]

PRACTICE EXERCISE 6.7

Repeat Example 6.7 for $V(r, \theta, \phi)$.

Answer: If $V(r, \theta, \phi) = R(r)\,F(\theta)\,\Phi(\phi)$, $\Phi'' + \lambda^2\Phi = 0$, $R'' + \frac{2}{r}R' - \frac{\mu^2}{r^2}R = 0$, $F'' + \cot\theta\, F' + (\mu^2 - \lambda^2 \operatorname{cosec}^2\theta)\,F = 0$.

6.5 RESISTANCE AND CAPACITANCE

In Section 5.4 the concept of resistance was covered and we derived eq. (5.16) for finding the resistance of a conductor of uniform cross section. If the cross section of the conductor is not uniform, eq. (5.16) becomes invalid and the resistance is obtained from eq. (5.17):

$$\boxed{R = \frac{V}{I} = \frac{\int \mathbf{E} \cdot d\mathbf{l}}{\oint \sigma\mathbf{E} \cdot d\mathbf{S}}} \tag{6.16}$$

The problem of finding the resistance of a conductor of nonuniform cross section can be treated as a boundary-value problem. Using eq. (6.16), the resistance R (or conductance $G = 1/R$) of a given conducting material can be found by following these steps:

1. Choose a suitable coordinate system.
2. Assume V_o as the potential difference between conductor terminals.
3. Solve Laplace's equation $\nabla^2 V$ to obtain V. Then determine $\mathbf{E}$ from $\mathbf{E} = -\nabla V$ and I from $I = \int \sigma\mathbf{E} \cdot d\mathbf{S}$.
4. Finally, obtain R as V_o/I.

In essence, we assume V_o, find I, and determine $R = V_o/I$. Alternatively, it is possible to assume current I_o, find the corresponding potential difference V, and determine R from $R = V/I_o$. As will be discussed shortly, the capacitance of a capacitor is obtained using a similar technique.

[3]For a complete solution of Laplace's equation in cylindrical or spherical coordinates, see, for example, D. T. Paris and F. K. Hurd, *Basic Electromagnetic Theory*. New York: McGraw-Hill, 1969, pp. 150–159.

Generally speaking, to have a capacitor we must have two (or more) conductors carrying equal but opposite charges. This implies that all the flux lines leaving one conductor must necessarily terminate at the surface of the other conductor. The conductors are sometimes referred to as the *plates* of the capacitor. The plates may be separated by free space or a dielectric.

Consider the two-conductor capacitor of Figure 6.12. The conductors are maintained at a potential difference V given by

$$V = V_1 - V_2 = -\int_2^1 \mathbf{E} \cdot d\mathbf{l} \tag{6.17}$$

where $\mathbf{E}$ is the electric field existing between the conductors and conductor 1 is assumed to carry a positive charge. (Note that the $\mathbf{E}$ field is always normal to the conducting surfaces.)

We define the *capacitance* C of the capacitor as the ratio of the magnitude of the charge on one of the plates to the potential difference between them; that is,

$$\boxed{C = \frac{Q}{V} = \frac{\varepsilon \oint \mathbf{E} \cdot d\mathbf{S}}{\int \mathbf{E} \cdot d\mathbf{l}}} \tag{6.18}$$

The negative sign before $V = -\int \mathbf{E} \cdot d\mathbf{l}$ has been dropped because we are interested in the absolute value of V. The capacitance C is a physical property of the capacitor and in measured in farads (F). Using eq. (6.18), C can be obtained for any given two-conductor capacitance by following either of these methods:

1. Assuming Q and determining V in terms of Q (involving Gauss's law)
2. Assuming V and determining Q in terms of V (involving solving Laplace's equation)

We shall use the former method here, and the latter method will be illustrated in Examples 6.10 and 6.11. The former method involves taking the following steps:

1. Choose a suitable coordinate system.
2. Let the two conducting plates carry charges $+Q$ and $-Q$.

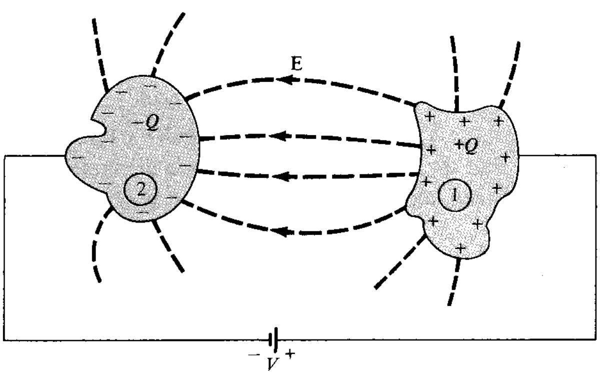

Figure 6.12 A two-conductor capacitor.

3. Determine **E** using Coulomb's or Gauss's law and find V from $V = -\int \mathbf{E} \cdot d\mathbf{l}$. The negative sign may be ignored in this case because we are interested in the absolute value of V.
4. Finally, obtain C from $C = Q/V$.

We will now apply this mathematically attractive procedure to determine the capacitance of some important two-conductor configurations.

A. Parallel-Plate Capacitor

Consider the parallel-plate capacitor of Figure 6.13(a). Suppose that each of the plates has an area S and they are separated by a distance d. We assume that plates 1 and 2, respectively, carry charges $+Q$ and $-Q$ uniformly distributed on them so that

$$\rho_S = \frac{Q}{S} \tag{6.19}$$

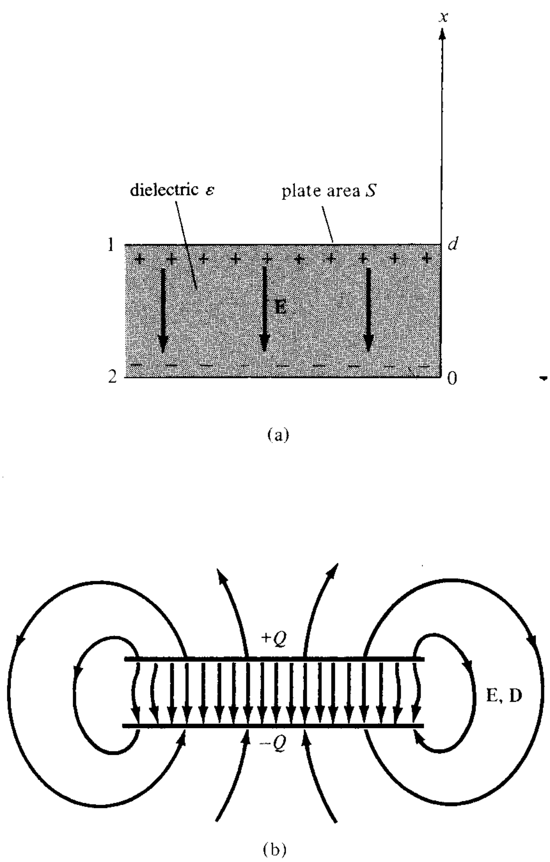

Figure 6.13 **(a)** Parallel-plate capacitor, **(b)** fringing effect due to a parallel-plate capacitor.

An ideal parallel-plate capacitor is one in which the plate separation d is very small compared with the dimensions of the plate. Assuming such an ideal case, the fringing field at the edge of the plates, as illustrated in Figure 6.13(b), can be ignored so that the field between them is considered uniform. If the space between the plates is filled with a homogeneous dielectric with permittivity ε and we ignore flux fringing at the edges of the plates, from eq. (4.27), $\mathbf{D} = -\rho_S \mathbf{a}_x$ or

$$\begin{aligned} \mathbf{E} &= \frac{\rho_S}{\varepsilon}(-\mathbf{a}_x) \\ &= -\frac{Q}{\varepsilon S}\mathbf{a}_x \end{aligned} \tag{6.20}$$

Hence

$$V = -\int_2^1 \mathbf{E} \cdot d\mathbf{l} = -\int_0^d \left[-\frac{Q}{\varepsilon S}\mathbf{a}_x\right] \cdot dx\, \mathbf{a}_x = \frac{Qd}{\varepsilon S} \tag{6.21}$$

and thus for a parallel-plate capacitor

$$\boxed{C = \frac{Q}{V} = \frac{\varepsilon S}{d}} \tag{6.22}$$

This formula offers a means of measuring the dielectric constant ε_r of a given dielectric. By measuring the capacitance C of a parallel-plate capacitor with the space between the plates filled with the dielectric and the capacitance C_o with air between the plates, we find ε_r from

$$\varepsilon_r = \frac{C}{C_o} \tag{6.23}$$

Using eq. (4.96), it can be shown that the energy stored in a capacitor is given by

$$\boxed{W_E = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{Q^2}{2C}} \tag{6.24}$$

To verify this for a parallel-plate capacitor, we substitute eq. (6.20) into eq. (4.96) and obtain

$$\begin{aligned} W_E &= \frac{1}{2}\int_v \varepsilon \frac{Q^2}{\varepsilon^2 S^2}\, dv = \frac{\varepsilon Q^2 S d}{2\varepsilon^2 S^2} \\ &= \frac{Q^2}{2}\left(\frac{d}{\varepsilon S}\right) = \frac{Q^2}{2C} = \frac{1}{2}QV \end{aligned}$$

as expected.

B. Coaxial Capacitor

This is essentially a coaxial cable or coaxial cylindrical capacitor. Consider length L of two coaxial conductors of inner radius a and outer radius b ($b > a$) as shown in Figure 6.14. Let the space between the conductors be filled with a homogeneous dielectric with permittivity ε. We assume that conductors 1 and 2, respectively, carry $+Q$ and $-Q$ uniformly distributed on them. By applying Gauss's law to an arbitrary Gaussian cylindrical surface of radius ρ ($a < \rho < b$), we obtain

$$Q = \varepsilon \oint \mathbf{E} \cdot d\mathbf{S} = \varepsilon E_\rho 2\pi\rho L \tag{6.25}$$

Hence:

$$\mathbf{E} = \frac{Q}{2\pi\varepsilon\rho L}\,\mathbf{a}_\rho \tag{6.26}$$

Neglecting flux fringing at the cylinder ends,

$$V = -\int_2^1 \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \left[\frac{Q}{2\pi\varepsilon\rho L}\,\mathbf{a}_\rho\right] \cdot d\rho\,\mathbf{a}_\rho \tag{6.27a}$$

$$= \frac{Q}{2\pi\varepsilon L}\ln\frac{b}{a} \tag{6.27b}$$

Thus the capacitance of a coaxial cylinder is given by

$$\boxed{C = \frac{Q}{V} = \frac{2\pi\varepsilon L}{\ln\dfrac{b}{a}}} \tag{6.28}$$

C. Spherical Capacitor

This is the case of two concentric spherical conductors. Consider the inner sphere of radius a and outer sphere of radius b ($b > a$) separated by a dielectric medium with permittivity ε as shown in Figure 6.15. We assume charges $+Q$ and $-Q$ on the inner and outer spheres

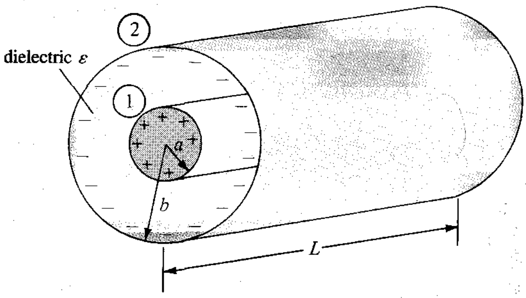

Figure 6.14 Coaxial capacitor.

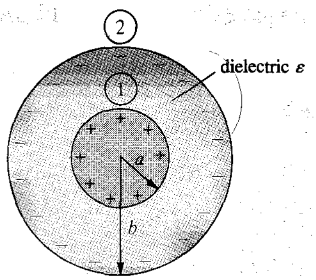

Figure 6.15 Spherical capacitor.

respectively. By applying Gauss's law to an arbitrary Gaussian spherical surface of radius $r\ (a < r < b)$,

$$Q = \varepsilon \oint \mathbf{E} \cdot d\mathbf{S} = \varepsilon E_r 4\pi r^2 \tag{6.29}$$

that is,

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon r^2}\,\mathbf{a}_r \tag{6.30}$$

The potential difference between the conductors is

$$\begin{aligned} V &= -\int_2^1 \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \left[\frac{Q}{4\pi\varepsilon r^2}\,\mathbf{a}_r\right] \cdot dr\,\mathbf{a}_r \\ &= \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right] \end{aligned} \tag{6.31}$$

Thus the capacitance of the spherical capacitor is

$$\boxed{C = \frac{Q}{V} = \frac{4\pi\varepsilon}{\dfrac{1}{a} - \dfrac{1}{b}}} \tag{6.32}$$

By letting $b \to \infty$, $C = 4\pi\varepsilon a$, which is the capacitance of a spherical capacitor whose outer plate is infinitely large. Such is the case of a spherical conductor at a large distance from other conducting bodies—the *isolated sphere*. Even an irregularly shaped object of about the same size as the sphere will have nearly the same capacitance. This fact is useful in estimating the stray capacitance of an isolated body or piece of equipment.

Recall from network theory that if two capacitors with capacitance C_1 and C_2 are in series (i.e., they have the same charge on them) as shown in Figure 6.16(a), the total capacitance is

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

or

$$C = \frac{C_1 C_2}{C_1 + C_2} \tag{6.33}$$

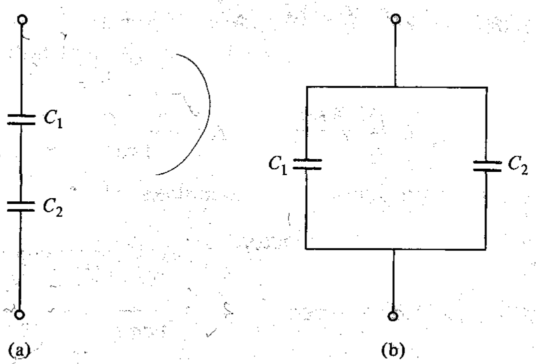

Figure 6.16 Capacitors in **(a)** series, and **(b)** parallel.

If the capacitors are in parallel (i.e., they have the same voltage across their plates) as shown in Figure 6.16(b), the total capacitance is

$$C = C_1 + C_2 \tag{6.34}$$

Let us reconsider the expressions for finding the resistance R and the capacitance C of an electrical system. The expressions were given in eqs. (6.16) and (6.18):

$$R = \frac{V}{I} = \frac{\int \mathbf{E} \cdot d\mathbf{l}}{\oint \sigma \mathbf{E} \cdot d\mathbf{S}} \tag{6.16}$$

$$C = \frac{Q}{V} = \frac{\varepsilon \oint \mathbf{E} \cdot d\mathbf{S}}{\int \mathbf{E} \cdot d\mathbf{l}} \tag{6.18}$$

The product of these expressions yields

$$\boxed{RC = \frac{\varepsilon}{\sigma}} \tag{6.35}$$

which is the relaxation time T_r of the medium separating the conductors. It should be remarked that eq. (6.35) is valid only when the medium is homogeneous; this is easily inferred from eqs. (6.16) and (6.18). Assuming homogeneous media, the resistance of various capacitors mentioned earlier can be readily obtained using eq. (6.35). The following examples are provided to illustrate this idea.

For a parallel-plate capacitor,

$$C = \frac{\varepsilon S}{d}, \qquad R = \frac{d}{\sigma S} \tag{6.36}$$

For a cylindrical capacitor,

$$C = \frac{2\pi\varepsilon L}{\ln \dfrac{b}{a}}, \qquad R = \frac{\ln \dfrac{b}{a}}{2\pi\sigma L} \tag{6.37}$$

For a spherical capacitor,

$$C = \frac{4\pi\varepsilon}{\dfrac{1}{a} - \dfrac{1}{b}}, \qquad R = \frac{\dfrac{1}{a} - \dfrac{1}{b}}{4\pi\sigma} \tag{6.38}$$

And finally for an isolated spherical conductor,

$$C = 4\pi\varepsilon a, \qquad R = \frac{1}{4\pi\sigma a} \tag{6.39}$$

It should be noted that the resistance R in each of eqs. (6.35) to (6.39) is not the resistance of the capacitor plate but the leakage resistance between the plates; therefore, σ in those equations is the conductivity of the dielectric medium separating the plates.

EXAMPLE 6.8

A metal bar of conductivity σ is bent to form a flat 90° sector of inner radius a, outer radius b, and thickness t as shown in Figure 6.17. Show that (a) the resistance of the bar between the vertical curved surfaces at $\rho = a$ and $\rho = b$ is

$$R = \frac{2 \ln \dfrac{b}{a}}{\sigma \pi t}$$

and (b) the resistance between the two horizontal surfaces at $z = 0$ and $z = t$ is

$$R' = \frac{4t}{\sigma\pi(b^2 - a^2)}$$

Solution:

(a) Between the vertical curved ends located at $\rho = a$ and $\rho = b$, the bar has a nonuniform cross section and hence eq. (5.16) does not apply. We have to use eq. (6.16). Let a potential difference V_o be maintained between the curved surfaces at $\rho = a$ and $\rho = b$ so that

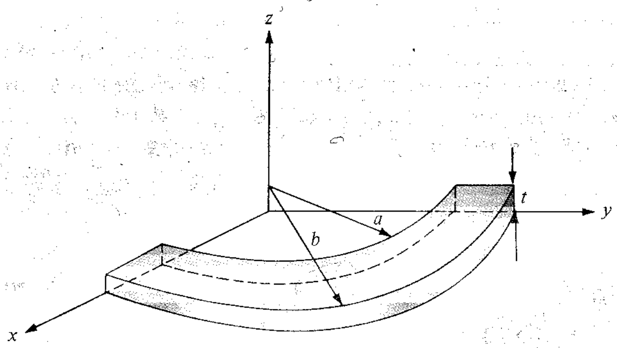

Figure 6.17 Metal bar of Example 6.8.

$V(\rho = a) = 0$ and $V(\rho = b) = V_o$. We solve for V in Laplace's equation $\nabla^2 V = 0$ in cylindrical coordinates. Since $V = V(\rho)$,

$$\nabla^2 V = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0$$

As $\rho = 0$ is excluded, upon multiplying by ρ and integrating once, this becomes

$$\rho \frac{dV}{d\rho} = A$$

or

$$\frac{dV}{d\rho} = \frac{A}{\rho}$$

Integrating once again yields

$$V = A \ln \rho + B$$

where A and B are constants of integration to be determined from the boundary conditions.

$$V(\rho = a) = 0 \rightarrow 0 = A \ln a + B \qquad \text{or} \qquad B = -A \ln a$$

$$V(\rho = b) = V_o \rightarrow V_o = A \ln b + B = A \ln b - A \ln a = A \ln \frac{b}{a} \qquad \text{or} \qquad A = \frac{V_o}{\ln \frac{b}{a}}$$

Hence,

$$V = A \ln \rho - A \ln a = A \ln \frac{\rho}{a} = \frac{V_o}{\ln \frac{b}{a}} \ln \frac{\rho}{a}$$

$$\mathbf{E} = -\nabla V = -\frac{dV}{d\rho}\mathbf{a}_\rho = -\frac{A}{\rho}\mathbf{a}_\rho = -\frac{V_o}{\rho \ln \frac{b}{a}}\mathbf{a}_\rho$$

$$\mathbf{J} = \sigma \mathbf{E}, \qquad d\mathbf{S} = -\rho \, d\phi \, dz \, \mathbf{a}_\rho$$

$$I = \int \mathbf{J} \cdot d\mathbf{S} = \int_{\phi=0}^{\pi/2} \int_{z=0}^{t} \frac{V_o \sigma}{\rho \ln \frac{b}{a}} \, dz \, \rho \, d\phi = \frac{\pi}{2} \frac{t V_o \sigma}{\ln \frac{b}{a}}$$

Thus

$$R = \frac{V_o}{I} = \frac{2 \ln \frac{b}{a}}{\sigma \pi t}$$

as required.

(b) Let V_o be the potential difference between the two horizontal surfaces so that $V(z = 0) = 0$ and $V(z = t) = V_o$. $V = V(z)$, so Laplace's equation $\nabla^2 V = 0$ becomes

$$\frac{d^2V}{dz^2} = 0$$

Integrating twice gives

$$V = Az + B$$

We apply the boundary conditions to determine A and B:

$$V(z = 0) = 0 \rightarrow 0 = 0 + B \qquad \text{or} \qquad B = 0$$

$$V(z = t) = V_o \rightarrow V_o = At \qquad \text{or} \qquad A = \frac{V_o}{t}$$

Hence,

$$V = \frac{V_o}{t} z$$

$$\mathbf{E} = -\nabla V = -\frac{dV}{dz}\mathbf{a}_z = -\frac{V_o}{t}\mathbf{a}_z$$

$$\mathbf{J} = \sigma\mathbf{E} = -\frac{\sigma V_o}{t}\mathbf{a}_z, \qquad d\mathbf{S} = -\rho \, d\phi \, d\rho \, \mathbf{a}_z$$

$$I = \int \mathbf{J} \cdot d\mathbf{S} = \int_{\rho=a}^{b} \int_{\phi=0}^{\pi/2} \frac{V_o\sigma}{t} \rho \, d\phi \, d\rho$$

$$= \frac{V_o\sigma}{t} \cdot \frac{\pi}{2} \frac{\rho^2}{2}\bigg|_a^b = \frac{V_o \, \sigma \, \pi \, (b^2 - a^2)}{4t}$$

Thus

$$R' = \frac{V_o}{I} = \frac{4t}{\sigma\pi(b^2 - a^2)}$$

Alternatively, for this case, the cross section of the bar is uniform between the horizontal surfaces at $z = 0$ and $z = t$ and eq. (5.16) holds. Hence,

$$R' = \frac{\ell}{\sigma S} = \frac{t}{\sigma \dfrac{\pi}{4}(b^2 - a^2)}$$

$$= \frac{4t}{\sigma\pi(b^2 - a^2)}$$

as required.

PRACTICE EXERCISE 6.8

A disc of thickness t has radius b and a central hole of radius a. Taking the conductivity of the disc as σ, find the resistance between

(a) The hole and the rim of the disc

(b) The two flat sides of the disc

Answer: (a) $\dfrac{\ln b/a}{2\pi t\sigma}$, (b) $\dfrac{t}{\sigma\pi(b^2 - a^2)}$.

A coaxial cable contains an insulating material of conductivity σ. If the radius of the central wire is a and that of the sheath is b, show that the conductance of the cable per unit length is (see eq. (6.37))

$$G = \frac{2\pi\sigma}{\ln b/a}$$

Solution:

Consider length L of the coaxial cable as shown in Figure 6.14. Let V_o be the potential difference between the inner and outer conductors so that $V(\rho = a) = 0$ and $V(\rho = b) = V_o$ V and $\mathbf{E}$ can be found just as in part (a) of the last example. Hence:

$$\mathbf{J} = \sigma\mathbf{E} = \frac{-\sigma V_o}{\rho \ln b/a}\,\mathbf{a}_\rho, \qquad d\mathbf{S} = -\rho d\phi\, dz\,\mathbf{a}_\rho$$

$$\begin{aligned} I &= \int \mathbf{J}\cdot d\mathbf{S} = \int_{\phi=0}^{2\pi}\int_{z=0}^{L} \frac{V_o\sigma}{\rho \ln b/a}\,\rho\, dz\, d\phi \\ &= \frac{2\pi L\sigma V_o}{\ln b/a} \end{aligned}$$

The resistance per unit length is

$$R = \frac{V_o}{I}\cdot\frac{1}{L} = \frac{\ln b/a}{2\pi\sigma}$$

and the conductance per unit length is

$$G = \frac{1}{R} = \frac{2\pi\sigma}{\ln b/a}$$

as required.

PRACTICE EXERCISE 6.9

A coaxial cable contains an insulating material of conductivity σ_1 in its upper half and another material of conductivity σ_2 in its lower half (similar to the situation in Figure 6.19b). If the radius of the central wire is a and that of the sheath is b, show that the leakage resistance of length ℓ of the cable is

$$R = \frac{1}{\pi \ell (\sigma_1 + \sigma_2)} \ln \frac{b}{a}$$

Answer: Proof.

EXAMPLE 6.10

Conducting spherical shells with radii $a = 10$ cm and $b = 30$ cm are maintained at a potential difference of 100 V such that $V(r = b) = 0$ and $V(r = a) = 100$ V. Determine V and $\mathbf{E}$ in the region between the shells. If $\varepsilon_r = 2.5$ in the region, determine the total charge induced on the shells and the capacitance of the capacitor.

Solution:

Consider the spherical shells shown in Figure 6.18. V depends only on r and hence Laplace's equation becomes

$$\nabla^2 V = \frac{1}{r^2} \frac{d}{dr}\left[r^2 \frac{dV}{dr}\right] = 0$$

Since $r \neq 0$ in the region of interest, we multiply through by r^2 to obtain

$$\frac{d}{dr}\left[r^2 \frac{dV}{dr}\right] = 0$$

Integrating once gives

$$r^2 \frac{dV}{dr} = A$$

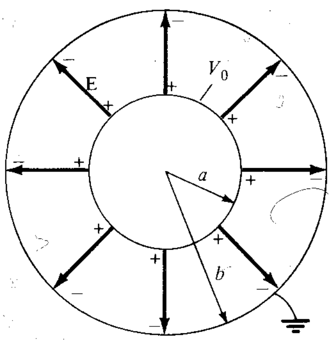

Figure 6.18 Potential $V(r)$ due to conducting spherical shells.

or

$$\frac{dV}{dr} = \frac{A}{r^2}$$

Integrating again gives

$$V = -\frac{A}{r} + B$$

As usual, constants A and B are determined from the boundary conditions.

When $r = b, V = 0 \rightarrow 0 = -\dfrac{A}{b} + B$ or $B = \dfrac{A}{b}$

Hence

$$V = A\left[\frac{1}{b} - \frac{1}{r}\right]$$

Also when $r = a, V = V_o \rightarrow V_o = A\left[\dfrac{1}{b} - \dfrac{1}{a}\right]$

or

$$A = \frac{V_o}{\dfrac{1}{b} - \dfrac{1}{a}}$$

Thus

$$V = V_o \frac{\left[\dfrac{1}{r} - \dfrac{1}{b}\right]}{\dfrac{1}{a} - \dfrac{1}{b}}$$

$$\mathbf{E} = -\nabla V = -\frac{dV}{dr}\mathbf{a}_r = -\frac{A}{r^2}\mathbf{a}_r$$

$$= \frac{V_o}{r^2\left[\dfrac{1}{a} - \dfrac{1}{b}\right]}\mathbf{a}_r$$

$$Q = \int \varepsilon \mathbf{E} \cdot d\mathbf{S} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\varepsilon_o \varepsilon_r V_o}{r^2\left[\dfrac{1}{a} - \dfrac{1}{b}\right]} r^2 \sin\theta \, d\phi \, d\theta$$

$$= \frac{4\pi\varepsilon_o\varepsilon_r V_o}{\dfrac{1}{a} - \dfrac{1}{b}}$$

The capacitance is easily determined as

$$C = \frac{Q}{V_o} = \frac{4\pi\varepsilon}{\dfrac{1}{a} - \dfrac{1}{b}}$$

which is the same as we obtained in eq. (6.32); there in Section 6.5, we assumed Q and found the corresponding V_o, but here we assumed V_o and found the corresponding Q to determine C. Substituting $a = 0.1$ m, $b = 0.3$ m, $V_o = 100$ V yields

$$V = 100 \frac{\left[\dfrac{1}{r} - \dfrac{10}{3}\right]}{10 - 10/3} = 15\left[\frac{1}{r} - \frac{10}{3}\right] V$$

Check: $\nabla^2 V = 0$, $V(r = 0.3 \text{ m}) = 0$, $V(r = 0.1 \text{ m}) = 100$.

$$\mathbf{E} = \frac{100}{r^2\,[10 - 10/3]}\,\mathbf{a}_r = \frac{15}{r^2}\,\mathbf{a}_r \text{ V/m}$$

$$\begin{aligned} Q &= \pm 4\pi \cdot \frac{10^{-9}}{36\pi} \cdot \frac{(2.5)\cdot(100)}{10 - 10/3} \\ &= \pm 4.167 \text{ nC} \end{aligned}$$

The positive charge is induced on the inner shell; the negative charge is induced on the outer shell. Also

$$C = \frac{|Q|}{V_o} = \frac{4.167 \times 10^{-9}}{100} = 41.67 \text{ pF}$$

PRACTICE EXERCISE 6.10

If Figure 6.19 represents the cross sections of two spherical capacitors, determine their capacitances. Let $a = 1$ mm, $b = 3$ mm, $c = 2$ mm, $\varepsilon_{r1} = 2.5$, and $\varepsilon_{r2} = 3.5$.

Answer: (a) 0.53 pF, (b) 0.5 pF

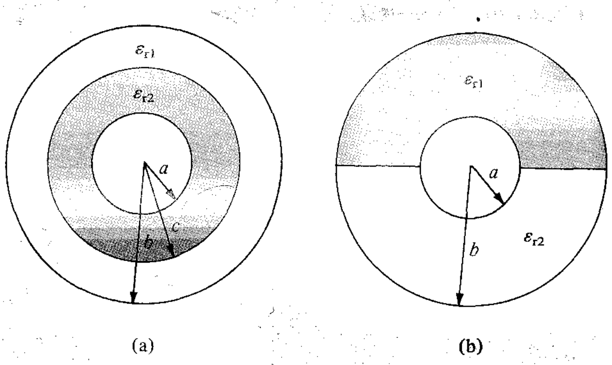

Figure 6.19 For Practice Exercises 6.9, 6.10, and 6.12.

EXAMPLE 6.11

In Section 6.5, it was mentioned that the capacitance $C = Q/V$ of a capacitor can be found by either assuming Q and finding V or by assuming V and finding Q. The former approach was used in Section 6.5 while we have used the latter method in the last example. Using the latter method, derive eq. (6.22).

Solution:

Assume that the parallel plates in Figure 6.13 are maintained at a potential difference V_o so that $V(x = 0)$ and $V(x = d) = V_o$. This necessitates solving a one-dimensional boundary-value problem; that is, we solve Laplace's equation

$$\nabla^2 V = \frac{d^2V}{dx^2} = 0$$

Integrating twice gives

$$V = Ax + B$$

where A and B are integration constants to be determined from the boundary conditions. At $x = 0$, $V = 0 \rightarrow 0 = 0 + B$, or $B = 0$, and at $x = d$, $V = V_o \rightarrow V_o = Ad + 0$ or $A = V_o/d$.

Hence

$$V = \frac{V_o}{d} x$$

Notice that this solution satisfies Laplace's equation and the boundary conditions.

We have assumed the potential difference between the plates to be V_o. Our goal is to find the charge Q on either plate so that we can eventually find the capacitance $C = Q/V_o$. The charge on either plate is

$$Q = \int \rho_S \, dS$$

But $\rho_S = \mathbf{D} \cdot \mathbf{a}_n = \varepsilon \mathbf{E} \cdot \mathbf{a}_n$, where

$$\mathbf{E} = -\nabla V = -\frac{dV}{dx}\mathbf{a}_x = -A\mathbf{a}_x = -\frac{V_o}{d}\mathbf{a}_x$$

On the lower plates, $\mathbf{a}_n = \mathbf{a}_x$, so

$$\rho_S = -\frac{\varepsilon V_o}{d} \quad \text{and} \quad Q = -\frac{\varepsilon V_o S}{d}$$

On the upper plates, $\mathbf{a}_n = -\mathbf{a}_x$, so

$$\rho_S = \frac{\varepsilon V_o}{d} \quad \text{and} \quad Q = \frac{\varepsilon V_o S}{d}$$

As expected, Q is equal but opposite on each plate. Thus

$$C = \frac{|Q|}{V_o} = \frac{\varepsilon S}{d}$$

which is in agreement with eq. (6.22).

PRACTICE EXERCISE 6.11

Derive the formula for the capacitance $C = Q/V_o$ of a cylindrical capacitor in eq. (6.28) by assuming V_o and finding Q.

EXAMPLE 6.12

Determine the capacitance of each of the capacitors in Figure 6.20. Take $\varepsilon_{r1} = 4$, $\varepsilon_{r2} = 6$, $d = 5$ mm, $S = 30\text{ cm}^2$.

Solution:

(a) Since **D** and **E** are normal to the dielectric interface, the capacitor in Figure 6.20(a) can be treated as consisting of two capacitors C_1 and C_2 in series as in Figure 6.16(a).

$$C_1 = \frac{\varepsilon_o \varepsilon_{r1} S}{d/2} = \frac{2\varepsilon_o \varepsilon_{r1} S}{d}, \qquad C_2 = \frac{2\varepsilon_o \varepsilon_{r2} S}{d}$$

The total capacitor C is given by

$$\begin{aligned} C &= \frac{C_1 C_2}{C_1 + C_2} = \frac{2\varepsilon_o S}{d} \frac{(\varepsilon_{r1}\varepsilon_{r2})}{\varepsilon_{r1} + \varepsilon_{r2}} \\ &= 2 \cdot \frac{10^{-9}}{36\pi} \cdot \frac{30 \times 10^{-4}}{5 \times 10^{-3}} \cdot \frac{4 \times 6}{10} \\ C &= 25.46 \text{ pF} \end{aligned} \tag{6.12.1}$$

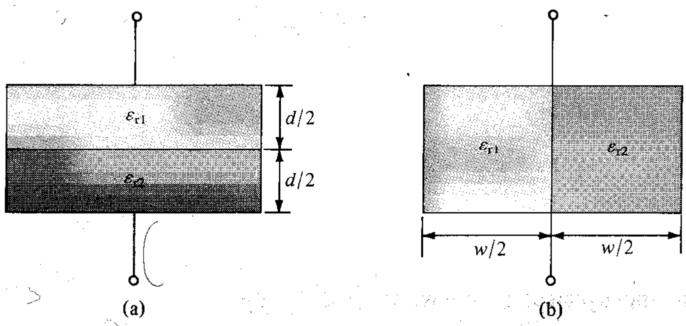

Figure 6.20 For Example 6.12.

(b) In this case, **D** and **E** are parallel to the dielectric interface. We may treat the capacitor as consisting of two capacitors C_1 and C_2 in parallel (the same voltage across C_1 and C_2) as in Figure 6.16(b).

$$C_1 = \frac{\varepsilon_o \varepsilon_{r1} S/2}{d} = \frac{\varepsilon_o \varepsilon_{r1} S}{2d}, \qquad C_2 = \frac{\varepsilon_o \varepsilon_{r2} S}{2d}$$

The total capacitance is

$$\begin{aligned} C &= C_1 + C_2 = \frac{\varepsilon_o S}{2d}(\varepsilon_{r1} + \varepsilon_{r2}) \\ &= \frac{10^{-9}}{36\pi} \cdot \frac{30 \times 10^{-4}}{2 \cdot (5 \times 10^{-3})} \cdot 10 \\ C &= 26.53 \text{ pF} \end{aligned} \tag{6.12.2}$$

Notice that when $\varepsilon_{r1} = \varepsilon_{r2} = \varepsilon_r$, eqs. (6.12.1) and (6.12.2) agree with eq. (6.22) as expected.

PRACTICE EXERCISE 6.12

Determine the capacitance of 10 m length of the cylindrical capacitors shown in Figure 6.19. Take $a = 1$ mm, $b = 3$ mm, $c = 2$ mm, $\varepsilon_{r1} = 2.5$, and $\varepsilon_{r2} = 3.5$.

Answer: (a) 1.41 nF, (b) 1.52 nF.

A cylindrical capacitor has radii $a = 1$ cm and $b = 2.5$ cm. If the space between the plates is filled with an inhomogeneous dielectric with $\varepsilon_r = (10 + \rho)/\rho$, where ρ is in centimeters, find the capacitance per meter of the capacitor.

Solution:

The procedure is the same as that taken in Section 6.5 except that eq. (6.27a) now becomes

$$\begin{aligned} V &= -\int_b^a \frac{Q}{2\pi\varepsilon_o \varepsilon_r \rho L}\, d\rho = -\frac{Q}{2\pi\varepsilon_o L}\int_b^a \frac{d\rho}{\rho\left(\dfrac{10+\rho}{\rho}\right)} \\ &= \frac{-Q}{2\pi\varepsilon_o L}\int_b^a \frac{d\rho}{10+\rho} = \frac{-Q}{2\pi\varepsilon_o L}\ln(10+\rho)\Big|_b^a \\ &= \frac{Q}{2\pi\varepsilon_o L}\ln\frac{10+b}{10+a} \end{aligned}$$

Thus the capacitance per meter is ($L = 1$ m)

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon_o}{\ln \dfrac{10 + b}{10 + a}} = 2\pi \cdot \frac{10^{-9}}{36\pi} \cdot \frac{1}{\ln \dfrac{12.5}{11.0}}$$

$$C = 434.6 \text{ pF/m}$$

PRACTICE EXERCISE 6.13

A spherical capacitor with $a = 1.5$ cm, $b = 4$ cm has an inhomogeneous dielectric of $\varepsilon = 10\varepsilon_o/r$. Calculate the capacitance of the capacitor.

Answer: 1.13 nF.

6.6 METHOD OF IMAGES

The method of images, introduced by Lord Kelvin in 1848, is commonly used to determine V, $\mathbf{E}$, $\mathbf{D}$, and ρ_S due to charges in the presence of conductors. By this method, we avoid solving Poisson's or Laplace's equation but rather utilize the fact that a conducting surface is an equipotential. Although the method does not apply to all electrostatic problems, it can reduce a formidable problem to a simple one.

theory states that a given charge configuration above an infinite conducting plane may be replaced by the charge configuration [illegible] an equipotential surface in place of the conducting plane.

Typical examples of point, line, and volume charge configurations are portrayed in Figure 6.21(a), and their corresponding image configurations are in Figure 6.21(b).

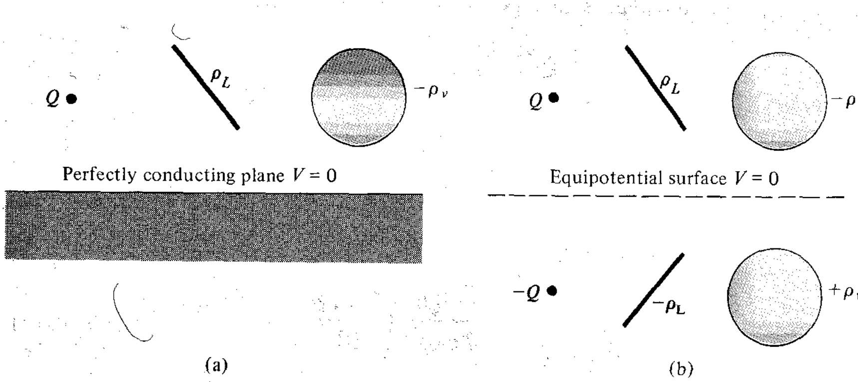

Figure 6.21 Image system: **(a)** charge configurations above a perfectly conducting plane; **(b)** image configuration with the conducting plane replaced by equipotential surface.

In applying the image method, two conditions must always be satisfied:

1. The image charge(s) must be located in the conducting region.
2. The image charge(s) must be located such that on the conducting surface(s) the potential is zero or constant.

The first condition is necessary to satisfy Poisson's equation, and the second condition ensures that the boundary conditions are satisfied. Let us now apply the image theory to two specific problems.

A. A Point Charge Above a Grounded Conducting Plane

Consider a point charge Q placed at a distance h from a perfect conducting plane of infinite extent as in Figure 6.22(a). The image configuration is in Figure 6.22(b). The electric field at point $P(x, y, z)$ is given by

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- \tag{6.40}$$

$$= \frac{Q\,\mathbf{r}_1}{4\pi\varepsilon_o r_1^3} + \frac{-Q\,\mathbf{r}_2}{4\pi\varepsilon_o r_2^3} \tag{6.41}$$

The distance vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are given by

$$\mathbf{r}_1 = (x, y, z) - (0, 0, h) = (x, y, z - h) \tag{6.42}$$

$$\mathbf{r}_2 = (x, y, z) - (0, 0, -h) = (x, y, z + h) \tag{6.43}$$

so eq. (6.41) becomes

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o}\left[\frac{x\mathbf{a}_x + y\mathbf{a}_y + (z - h)\mathbf{a}_z}{[x^2 + y^2 + (z - h)^2]^{3/2}} - \frac{x\mathbf{a}_x + y\mathbf{a}_y + (z + h)\mathbf{a}_z}{[x^2 + y^2 + (z + h)^2]^{3/2}}\right] \tag{6.44}$$

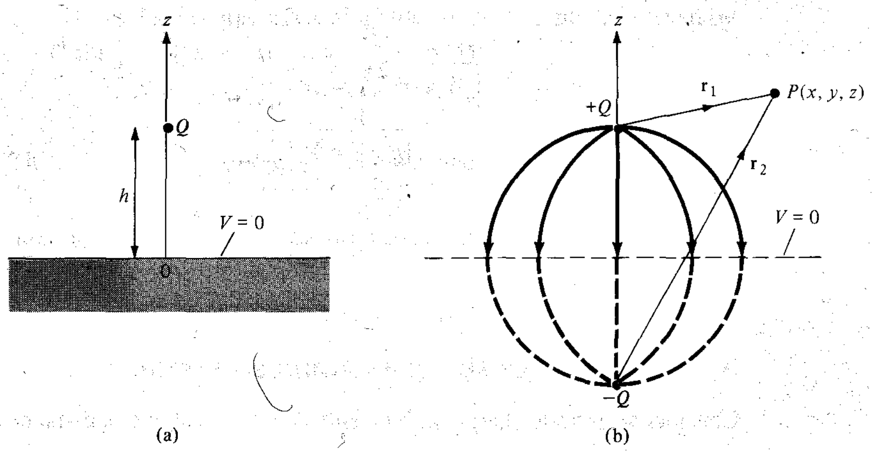

Figure 6.22 **(a)** Point charge and grounded conducting plane, **(b)** image configuration and field lines.

It should be noted that when $z = 0$, $\mathbf{E}$ has only the z-component, confirming that $\mathbf{E}$ is normal to the conducting surface.

The potential at P is easily obtained from eq. (6.41) or (6.44) using $V = -\int \mathbf{E} \cdot d\mathbf{l}$. Thus

$$\begin{aligned} V &= V_+ + V_- \\ &= \frac{Q}{4\pi\varepsilon_o r_1} + \frac{-Q}{4\pi\varepsilon_o r_2} \\ V &= \frac{Q}{4\pi\varepsilon_o}\left\{\frac{1}{[x^2 + y^2 + (z-h)^2]^{1/2}} - \frac{1}{[x^2 + y^2 + (z+h)^2]^{1/2}}\right\} \end{aligned} \tag{6.45}$$

for $z \geq 0$ and $V = 0$ for $z \leq 0$. Note that $V(z = 0) = 0$.

The surface charge density of the induced charge can also be obtained from eq. (6.44) as

$$\begin{aligned} \rho_S = D_n &= \varepsilon_o E_n \Big|_{z=0} \\ &= \frac{-Qh}{2\pi[x^2 + y^2 + h^2]^{3/2}} \end{aligned} \tag{6.46}$$

The total induced charge on the conducting plane is

$$Q_i = \int \rho_S \, dS = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{-Qh\,dx\,dy}{2\pi[x^2 + y^2 + h^2]^{3/2}} \tag{6.47}$$

By changing variables, $\rho^2 = x^2 + y^2$, $dx\,dy = \rho\,d\rho\,d\phi$.

$$Q_i = -\frac{Qh}{2\pi}\int_0^{2\pi}\int_0^{\infty} \frac{\rho\,d\rho\,d\phi}{[\rho^2 + h^2]^{3/2}} \tag{6.48}$$

Integrating over ϕ gives 2π, and letting $\rho\,d\rho = \frac{1}{2}d\,(\rho^2)$, we obtain

$$\begin{aligned} Q_i &= -\frac{Qh}{2\pi}2\pi \int_0^{\infty} [\rho^2 + h^2]^{-3/2}\frac{1}{2}\,d(\rho^2) \\ &= \frac{Qh}{[\rho^2 + h^2]^{1/2}}\Big|_0^{\infty} \\ &= -Q \end{aligned} \tag{6.49}$$

as expected, because all flux lines terminating on the conductor would have terminated on the image charge if the conductor were absent.

B. A Line Charge above a Grounded Conducting Plane

Consider an infinite charge with density ρ_L C/m located at a distance h from the grounded conducting plane $z = 0$. The same image system of Figure 6.22(b) applies to the line charge except that Q is replaced by ρ_L. The infinite line charge ρ_L may be assumed to be at

$x = 0, z = h$ and the image $-\rho_L$ at $x = 0, z = -h$ so that the two are parallel to the y-axis. The electric field at point P is given (from eq. 4.21) by

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- \tag{6.50}$$

$$= \frac{\rho_L}{2\pi\varepsilon_o\rho_1}\mathbf{a}_{\rho 1} + \frac{-\rho_L}{2\pi\varepsilon_o\rho_2}\mathbf{a}_{\rho 2} \tag{6.51}$$

The distance vectors $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are given by

$$\boldsymbol{\rho}_1 = (x, y, z) - (0, y, h) = (x, 0, z - h) \tag{6.52}$$

$$\boldsymbol{\rho}_2 = (x, y, z) - (0, y, -h) = (x, 0, z + h) \tag{6.53}$$

so eq. (6.51) becomes

$$\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_o}\left[\frac{x\mathbf{a}_x + (z - h)\mathbf{a}_z}{x^2 + (z - h)^2} - \frac{x\mathbf{a}_x + (z + h)\mathbf{a}_z}{x^2 + (z + h)^2}\right] \tag{6.54}$$

Again, notice that when $z = 0$, $\mathbf{E}$ has only the z-component, confirming that $\mathbf{E}$ is normal to the conducting surface.

The potential at P is obtained from eq. (6.51) or (6.54) using $V = -\int \mathbf{E} \cdot d\mathbf{l}$. Thus

$$\begin{aligned} V &= V_+ + V_- \\ &= -\frac{\rho_L}{2\pi\varepsilon_o}\ln\rho_1 - \frac{-\rho_L}{2\pi\varepsilon_o}\ln\rho_2 \\ &= -\frac{\rho_L}{2\pi\varepsilon_o}\ln\frac{\rho_1}{\rho_2} \end{aligned} \tag{6.55}$$

Substituting $\rho_1 = |\boldsymbol{\rho}_1|$ and $\rho_2 = |\boldsymbol{\rho}_2|$ in eqs. (6.52) and (6.53) into eq. (6.55) gives

$$V = -\frac{\rho_L}{2\pi\varepsilon_o}\ln\left[\frac{x^2 + (z - h)^2}{x^2 + (z + h)^2}\right]^{1/2} \tag{6.56}$$

for $z \geq 0$ and $V = 0$ for $z \leq 0$. Note that $V(z = 0) = 0$.

The surface charge induced on the conducting plane is given by

$$\rho_S = D_n = \varepsilon_o E_z\Big|_{z=0} = \frac{-\rho_L h}{\pi(x^2 + h^2)} \tag{6.57}$$

The induced charge per length on the conducting plane is

$$\rho_i = \int \rho_S\,dx = -\frac{\rho_L h}{\pi}\int_{-\infty}^{\infty}\frac{dx}{x^2 + h^2} \tag{6.58}$$

By letting $x = h\tan\alpha$, eq. (6.58) becomes

$$\begin{aligned} \rho_i &= -\frac{\rho_L h}{\pi}\int_{-\pi/2}^{\pi/2}\frac{d\alpha}{h} \\ &= -\rho_L \end{aligned} \tag{6.59}$$

as expected.

EXAMPLE 6.14

A point charge Q is located at point $(a, 0, b)$ between two semiinfinite conducting planes intersecting at right angles as in Figure 6.23. Determine the potential at point $P(x, y, z)$ and the force on Q.

Solution:

The image configuration is shown in Figure 6.24. Three image charges are necessary to satisfy the conditions in Section 6.6. From Figure 6.24(a), the potential at point $P(x, y, z)$ is the superposition of the potentials at P due to the four point charges; that is,

$$V = \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right]$$

where

$$r_1 = [(x - a)^2 + y^2 + (z - b)^2]^{1/2}$$
$$r_2 = [(x + a)^2 + y^2 + (z - b)^2]^{1/2}$$
$$r_3 = [(x + a)^2 + y^2 + (z + b)^2]^{1/2}$$
$$r_4 = [(x - a)^2 + y^2 + (z + b)^2]^{1/2}$$

From Figure 6.24(b), the net force on Q

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \\ &= -\frac{Q^2}{4\pi\varepsilon_o(2b)^2}\mathbf{a}_z - \frac{Q^2}{4\pi\varepsilon_o(2a)^2}\mathbf{a}_x + \frac{Q^2(2a\mathbf{a}_x + 2b\mathbf{a}_z)}{4\pi\varepsilon_o[(2a)^2 + (2b)^2]^{3/2}} \\ &= \frac{Q^2}{16\pi\varepsilon_o}\left\{\left[\frac{a}{(a^2 + b^2)^{3/2}} - \frac{1}{a^2}\right]\mathbf{a}_x + \left[\frac{b}{(a^2 + b^2)^{3/2}} - \frac{1}{b^2}\right]\mathbf{a}_z\right\}\end{aligned}$$

The electric field due to this system can be determined similarly and the charge induced on the planes can also be found.

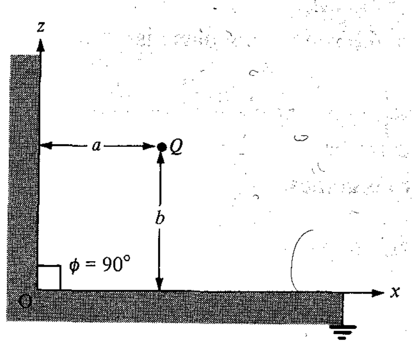

Figure 6.23 Point charge between two semiinfinite conducting planes.

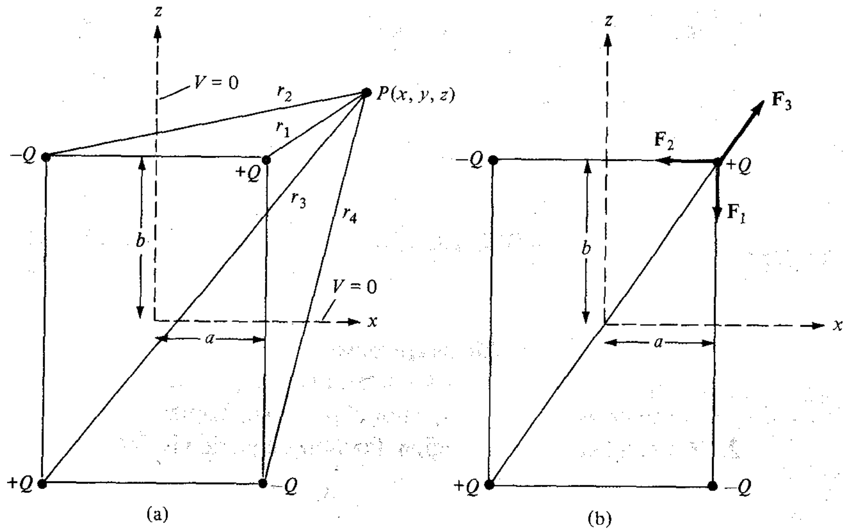

Figure 6.24 Determining **(a)** the potential at P, and **(b)** the force on charge Q.

In general, when the method of images is used for a system consisting of a point charge between two semiinfinite conducting planes inclined at an angle ϕ (in degrees), the number of images is given by

$$\boxed{N = \left(\frac{360°}{\phi} - 1\right)}$$

because the charge and its images all lie on a circle. For example, when $\phi = 180°$, $N = 1$ as in the case of Figure 6.22; for $\phi = 90°$, $N = 3$ as in the case of Figure 6.23; and for $\phi = 60°$, we expect $N = 5$ as shown in Figure 6.25.

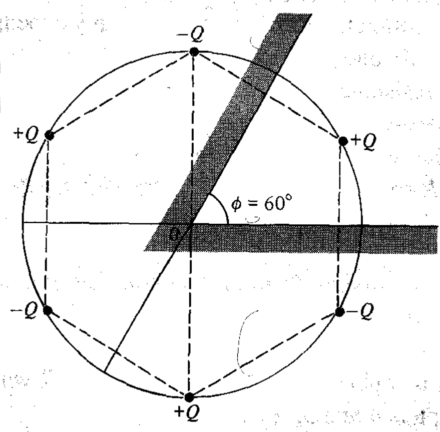

Figure 6.25 Point charge between two semiinfinite conducting walls inclined at $\phi = 60°$ to each.

PRACTICE EXERCISE 6.14

If the point charge $Q = 10$ nC in Figure 6.25 is 10 cm away from point O and along the line bisecting $\phi = 60°$, find the magnitude of the force on Q due to the charge induced on the conducting walls.

Answer: 60.53 μN.

SUMMARY

1. Boundary-value problems are those in which the potentials at the boundaries of a region are specified and we are to determine the potential field within the region. They are solved using Poisson's equation if $\rho_v \neq 0$ or Laplace's equation if $\rho_v = 0$.
2. In a nonhomogeneous region, Poisson's equation is

$$\nabla \cdot \varepsilon \nabla V = -\rho_v$$

For a homogeneous region, ε is independent of space variables. Poisson's equation becomes

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

In a charge-free region ($\rho_v = 0$), Poisson's equation becomes Laplace's equation; that is,

$$\nabla^2 V = 0$$

3. We solve the differential equation resulting from Poisson's or Laplace's equation by integrating twice if V depends on one variable or by the method of separation of variables if V is a function of more than one variable. We then apply the prescribed boundary conditions to obtain a unique solution.
4. The uniqueness theorem states that if V satisfies Poisson's or Laplace's equation and the prescribed boundary condition, V is the only possible solution for that given problem. This enables us to find the solution to a given problem via any expedient means because we are assured of one, and only one, solution.
5. The problem of finding the resistance R of an object or the capacitance C of a capacitor may be treated as a boundary-value problem. To determine R, we assume a potential difference V_o between the ends of the object, solve Laplace's equation, find $I = \int \sigma \mathbf{E} \cdot d\mathbf{S}$, and obtain $R = V_o/I$. Similarly, to determine C, we assume a potential difference of V_o between the plates of the capacitor, solve Laplace's equation, find $Q = \int \varepsilon \mathbf{E} \cdot d\mathbf{S}$, and obtain $C = Q/V_o$.
6. A boundary-value problem involving an infinite conducting plane or wedge may be solved using the method of images. This basically entails replacing the charge configuration by itself, its image, and an equipotential surface in place of the conducting plane. Thus the original problem is replaced by "an image problem," which is solved using techniques covered in Chapters 4 and 5.

REVIEW QUESTIONS

6.1 Equation $\nabla \cdot (-\varepsilon \nabla V) = \rho_v$ may be regarded as Poisson's equation for an inhomogeneous medium.

(a) True
(b) False

6.2 In cylindrical coordinates, equation

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} + 10 = 0$$

is called

(a) Maxwell's equation
(b) Laplace's equation
(c) Poisson's equation
(d) Helmholtz's equation
(e) Lorentz's equation

6.3 Two potential functions V_1 and V_2 satisfy Laplace's equation within a closed region and assume the same values on its surface. V_1 must be equal to V_2.

(a) True
(b) False
(c) Not necessarily

6.4 Which of the following potentials does not satisfy Laplace's equation?

(a) $V = 2x + 5$
(b) $V = 10\,xy$
(c) $V = r \cos \phi$
(d) $V = \dfrac{10}{r}$
(e) $V = \rho \cos \phi + 10$

6.5 Which of the following is not true?

(a) $-5 \cos 3x$ is a solution to $\phi''(x) + 9\phi(x) = 0$
(b) $10 \sin 2x$ is a solution to $\phi''(x) - 4\phi(x) = 0$
(c) $-4 \cosh 3y$ is a solution to $R''(y) - 9R(y) = 0$
(d) $\sinh 2y$ is a solution to $R''(y) - 4R(y) = 0$
(e) $\dfrac{g''(x)}{g(x)} = -\dfrac{h''(y)}{h(y)} = f(z) = -1$ where $g(x) = \sin x$ and $h(y) = \sinh y$

6.6 If $V_1 = X_1 Y_1$ is a product solution of Laplace's equation, which of these are not solutions of Laplace's equation?

(a) $-10X_1Y_1$
(b) $X_1Y_1 + 2xy$
(c) $X_1Y_1 - x + y$
(d) $X_1 + Y_1$
(e) $(X_1 - 2)(Y_1 + 3)$

6.7 The capacitance of a capacitor filled by a linear dielectric is independent of the charge on the plates and the potential difference between the plates.

(a) True
(b) False

6.8 A parallel-plate capacitor connected to a battery stores twice as much charge with a given dielectric as it does with air as dielectric, the susceptibility of the dielectric is

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4

6.9 A potential difference V_o is applied to a mercury column in a cylindrical container. The mercury is now poured into another cylindrical container of half the radius and the same potential difference V_o applied across the ends. As a result of this change of space, the resistance will be increased

(a) 2 times
(b) 4 times
(c) 8 times
(d) 16 times

6.10 Two conducting plates are inclined at an angle 30° to each other with a point charge between them. The number of image charges is

(a) 12
(b) 11
(c) 6
(d) 5
(e) 3

Answers: 6.1a, 6.2c, 6.3a, 6.4c, 6.5b, 6.6d,e, 6.7a, 6.8b, 6.9d, 6.10b.

PROBLEMS

6.1 In free space, $V = 6xy^2z + 8$. At point $P(1, 2, -5)$, find $\mathbf{E}$ and ρ_v.

6.2 Two infinitely large conducting plates are located at $x = 1$ and $x = 4$. The space between them is free space with charge distribution $\dfrac{x}{6\pi}$ nC/m^3. Find V at $x = 2$ if $V(1) = -50$ V and $V(4) = 50$ V.

6.3 The region between $x = 0$ and $x = d$ is free space and has $\rho_v = \rho_o(x - d)/d$. If $V(x = 0) = 0$ and $V(x = d) = V_o$, find: (a) V and $\mathbf{E}$, (b) the surface charge densities at $x = 0$ and $x = d$.

6.4 Show that the exact solution of the equation

$$\frac{d^2V}{dx^2} = f(x) \qquad 0 < x < L$$

subject to

$$V(x = 0) = V_1 \qquad V(x = L) = V_2$$

is

$$V(x) = \left[V_2 - V_1 - \int_0^L \int_0^\lambda f(\mu)\, d\mu\, d\lambda\right]\frac{x}{L} + V_1 + \int_0^x \int_0^\lambda f(\mu)\, d\mu\, d\lambda$$

6.5 A certain material occupies the space between two conducting slabs located at $y = \pm 2$ cm. When heated, the material emits electrons such that $\rho_v = 50(1 - y^2)\ \mu$C/m^3. If the slabs are both held at 30 kV, find the potential distribution within the slabs. Take $\varepsilon = 3\varepsilon_o$.

6.6 Determine which of the following potential field distributions satisfy Laplace's equation.

(a) $V_1 = x^2 + y^2 - 2z^2 + 10$

(b) $V_2 = \dfrac{1}{(x^2 + y^2 + z^2)^{1/2}}$

(c) $V_3 = \rho z \sin\phi + \rho^2$

(d) $V_4 = \dfrac{10 \sin\theta \sin\phi}{r^2}$

6.7 Show that the following potentials satisfy Laplace's equation.

(a) $V = e^{-5x} \cos 13y \sinh 12z$

(b) $V = \dfrac{z \cos\phi}{\rho}$

(c) $V = \dfrac{30 \cos\theta}{r^2}$

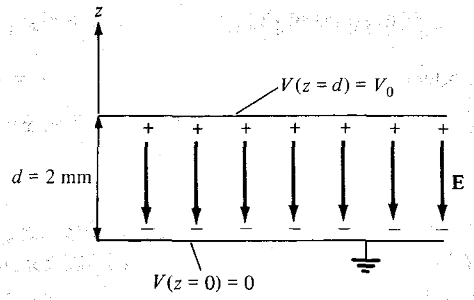

Figure 6.26 For Problem 6.11.

6.8 Show that $\mathbf{E} = (E_x, E_y, E_z)$ satisfies Laplace's equation.

6.9 Let $V = (A \cos nx + B \sin nx)(Ce^{ny} + De^{-ny})$, where A, B, C, and D are constants. Show that V satisfies Laplace's equation.

6.10 The potential field $V = 2x^2yz - y^3z$ exists in a dielectric medium having $\varepsilon = 2\varepsilon_o$. (a) Does V satisfy Laplace's equation? (b) Calculate the total charge within the unit cube $0 < x,y,z < 1$ m.

6.11 Consider the conducting plates shown in Figure 6.26. If $V(z = 0) = 0$ and $V(z = 2 \text{ mm}) = 50$ V, determine V, $\mathbf{E}$, and $\mathbf{D}$ in the dielectric region ($\varepsilon_r = 1.5$) between the plates and ρ_S on the plates.

6.12 The cylindrical capacitor whose cross section is in Figure 6.27 has inner and outer radii of 5 mm and 15 mm, respectively. If $V(\rho = 5 \text{ mm}) = 100$ V and $V(\rho = 15 \text{ mm}) = 0$ V, calculate V, $\mathbf{E}$, and $\mathbf{D}$ at $\rho = 10$ mm and ρ_S on each plate. Take $\varepsilon_r = 2.0$.

6.13 Concentric cylinders $\rho = 2$ cm and $\rho = 6$ cm are maintained at $V = 60$ V and $V = -20$ V, respectively. Calculate V, $\mathbf{E}$, and $\mathbf{D}$ at $\rho = 4$ cm.

6.14 The region between concentric spherical conducting shells $r = 0.5$ m and $r = 1$ m is charge free. If $V(r = 0.5) = -50$ V and $V(r = 1) = 50$ V, determine the potential distribution and the electric field strength in the region between the shells.

6.15 Find V and $\mathbf{E}$ at (3, 0, 4) due to the two conducting cones of infinite extent shown in Figure 6.28.

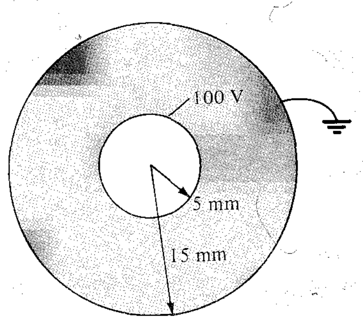

Figure 6.27 Cylindrical capacitor of Problem 6.12.

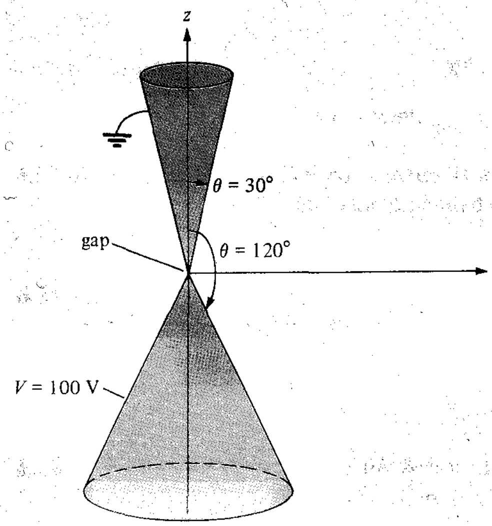

Figure 6.28 Conducting cones of Problem 6.15.

***6.16** The inner and outer electrodes of a diode are coaxial cylinders of radii $a = 0.6$ m and $b = 30$ mm, respectively. The inner electrode is maintained at 70 V while the outer electrode is grounded. (a) Assuming that the length of the electrodes $\ell \gg a, b$ and ignoring the effects of space charge, calculate the potential at $\rho = 15$ mm. (b) If an electron is injected radially through a small hole in the inner electrode with velocity 10^7 m/s, find its velocity at $\rho = 15$ mm.

6.17 Another method of finding the capacitance of a capacitor is using energy considerations, that is

$$C = \frac{2W_E}{V_o^2} = \frac{1}{V_o^2} \int \varepsilon |\mathbf{E}|^2 \, dv$$

Using this approach, derive eqs. (6.22), (6.28), and (6.32).

6.18 An electrode with a hyperbolic shape ($xy = 4$) is placed above an earthed right-angle corner as in Figure 6.29. Calculate V and $\mathbf{E}$ at point (1, 2, 0) when the electrode is connected to a 20-V source.

***6.19** Solve Laplace's equation for the two-dimensional electrostatic systems of Figure 6.30 and find the potential $V(x, y)$.

***6.20** Find the potential $V(x, y)$ due to the two-dimensional systems of Figure 6.31.

6.21 By letting $V(\rho, \phi) = R(\rho)\Phi(\phi)$ be the solution of Laplace's equation in a region where $\rho \neq 0$, show that the separated differential equations for R and Φ are

$$R'' + \frac{R'}{\rho} - \frac{\lambda}{\rho^2} R = 0$$

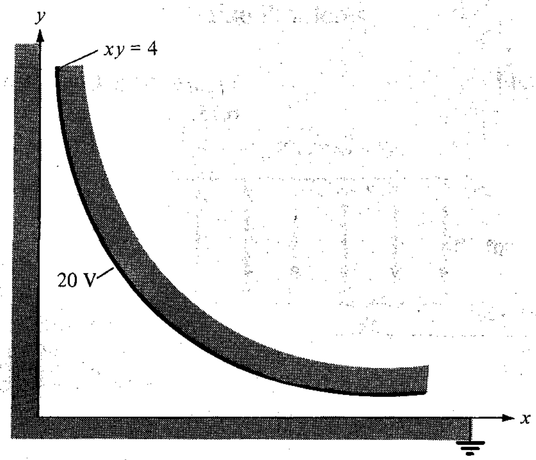

Figure 6.29 For Problem 6.18.

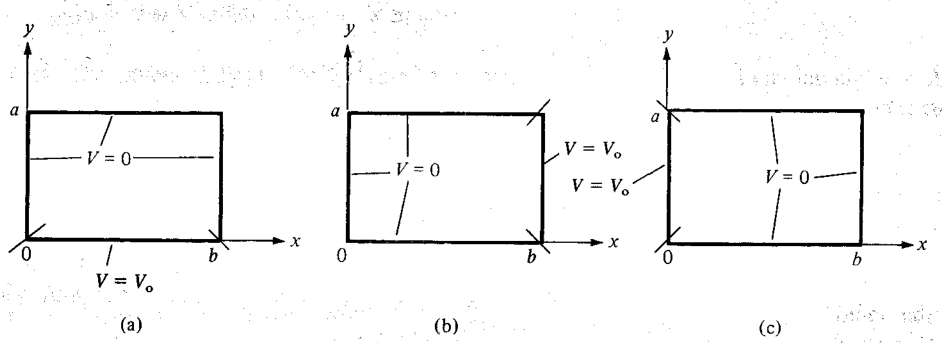

Figure 6.30 For Problem 6.19.

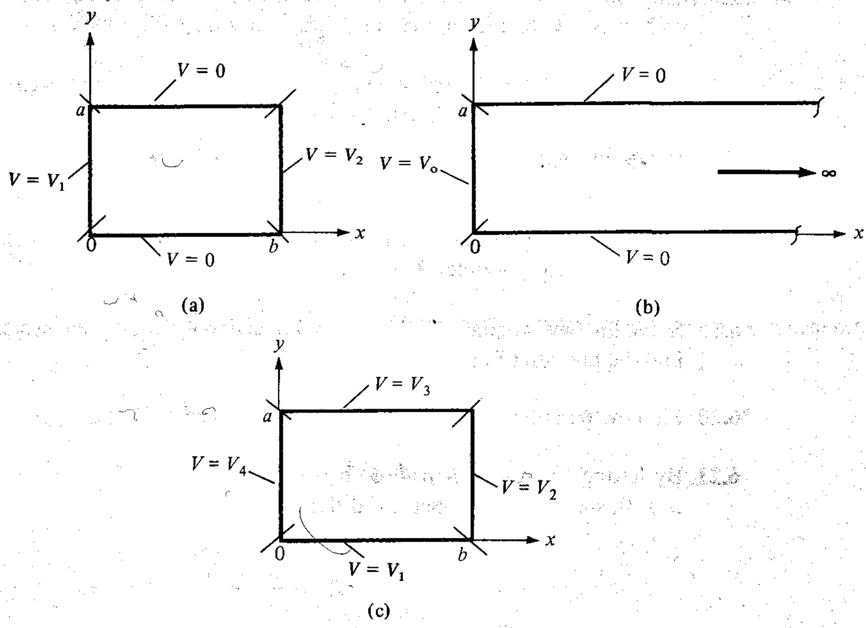

Figure 6.31 For Problem 6.20.

and

$$\Phi'' + \lambda\Phi = 0$$

where λ is the separation constant.

6.22 A potential in spherical coordinates is a function of r and θ but not ϕ. Assuming that $V(r, \theta) = R(r)F(\theta)$, obtain the separated differential equations for R and F in a region for which $\rho_v = 0$.

6.23 Show that the resistance of the bar of Figure 6.17 between the vertical ends located at $\phi = 0$ and $\phi = \pi/2$ is

$$R = \frac{\pi}{2\sigma t \ln b/a}$$

***6.24** Show that the resistance of the sector of a spherical shell of conductivity σ, with cross section shown in Figure 6.32 (where $0 \leq \phi < 2\pi$), between its base is

$$R = \frac{1}{2\pi\sigma(1 - \cos\alpha)}\left[\frac{1}{a} - \frac{1}{b}\right]$$

***6.25** A hollow conducting hemisphere of radius a is buried with its flat face lying flush with the earth surface thereby serving as an earthing electrode. If the conductivity of earth is σ, show that the leakage conductance between the electrode and earth is $2\pi a\sigma$.

6.26 The cross section of an electric fuse is shown in Figure 6.33. If the fuse is made of copper and of thickness 1.5 mm, calculate its resistance.

6.27 In an integrated circuit, a capacitor is formed by growing a silicon dioxide layer ($\varepsilon_r = 4$) of thickness 1 μm over the conducting silicon substrate and covering it with a metal electrode of area S. Determine S if a capacitance of 2 nF is desired.

6.28 The parallel-plate capacitor of Figure 6.34 is quarter-filled with mica ($\varepsilon_r = 6$). Find the capacitance of the capacitor.

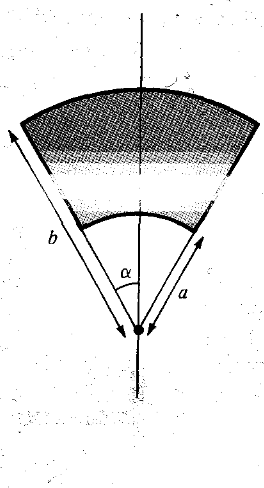

Figure 6.32 For Problem 6.24.

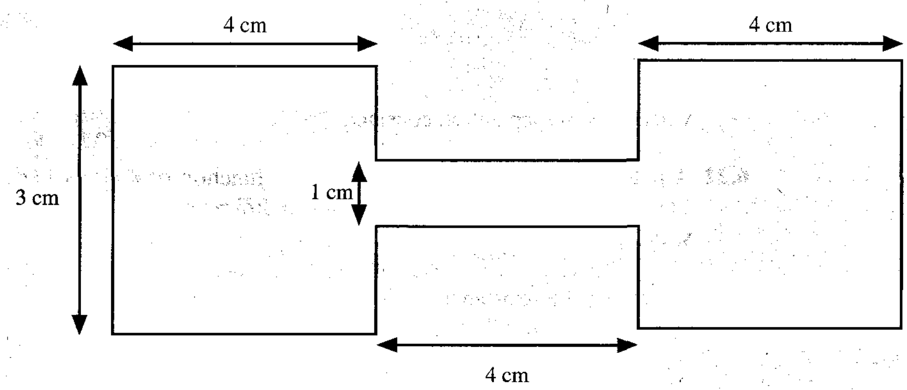

Figure 6.33 For Problem 6.26.

***6.29** An air-filled parallel plate capacitor of length L, width a, and plate separation d has its plates maintained at constant potential difference V_o. If a dielectric slab of dielectric constant ε_r is slid between the plates and is withdrawn until only a length x remains between the plates as in Figure 6.35, show that the force tending to restore the slab to its original position is

$$F = \frac{\varepsilon_o(\varepsilon_r - 1)\, a\, V_o^2}{2d}$$

6.30 A parallel-plate capacitor has plate area 200 cm^2 and plate separation 3 mm. The charge density is 1 μC/m^2 with air as dielectric. Find

(a) The capacitance of the capacitor
(b) The voltage between the plates
(c) The force with which the plates attract each other

6.31 Two conducting plates are placed at $z = -2$ cm and $z = 2$ cm and are, respectively, maintained at potentials 0 and 200 V. Assuming that the plates are separated by a polypropylene ($\varepsilon = 2.25\varepsilon_o$). Calculate: (a) the potential at the middle of the plates, (b) the surface charge densities at the plates.

6.32 Two conducting parallel plates are separated by a dielectric material with $\varepsilon = 5.6\varepsilon_o$ and thickness 0.64 mm. Assume that each plate has an area of 80 cm^2. If the potential field distribution between the plates is $V = 3x + 4y - 12z + 6$ kV, determine: (a) the capacitance of the capacitor, (b) the potential difference between the plates.

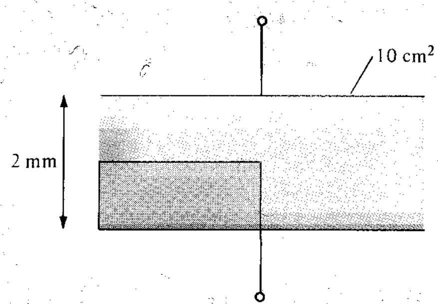

Figure 6.34 For Problem 6.28.

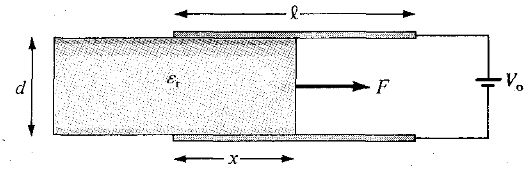

Figure 6.35 For Problem 6.29.

6.33 The space between spherical conducting shells $r = 5$ cm and $r = 10$ cm is filled with a dielectric material for which $\varepsilon = 2.25\varepsilon_o$. The two shells are maintained at a potential difference of 80 V. (a) Find the capacitance of the system. (b) Calculate the charge density on shell $r = 5$ cm.

6.34 Concentric shells $r = 20$ cm and $r = 30$ cm are held at $V = 0$ and $V = 50$, respectively. If the space between them is filled with dielectric material ($\varepsilon = 3.1\varepsilon_o$, $\sigma = 10^{-12}$ S/m), find: (a) V, **E**, and **D**, (b) the charge densities on the shells, (c) the leakage resistance.

6.35 A spherical capacitor has inner radius a and outer radius d. Concentric with the spherical conductors and lying between them is a spherical shell of outer radius c and inner radius b. If the regions $d < r < c$, $c < r < b$, and $b < r < a$ are filled with materials with permittivites ε_1, ε_2, and ε_3, respectively, determine the capacitance of the system.

6.36 Determine the capacitance of a conducting sphere of radius 5 cm deeply immersed in sea water ($\varepsilon_r = 80$).

6.37 A conducting sphere of radius 2 cm is surrounded by a concentric conducting sphere of radius 5 cm. If the space between the spheres is filled with sodium chloride ($\varepsilon_r = 5.9$), calculate the capacitance of the system.

***6.38** In an ink-jet printer the drops are charged by surrounding the jet of radius 20 μm with a concentric cylinder of radius 600 μm as in Figure 6.36. Calculate the minimum voltage required to generate a charge 50 fC on the drop if the length of the jet inside the cylinder is 100 μm. Take $\varepsilon = \varepsilon_o$.

6.39 A given length of a cable, the capacitance of which is 10 μF/km with a resistance of insulation of 100 MΩ/km, is charged to a voltage of 100 V. How long does it take the voltage to drop to 50 V?

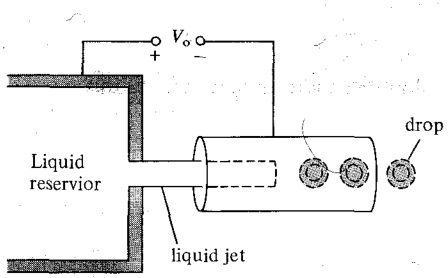

Figure 6.36 Simplified geometry of an ink-jet printer; for Problem 6.38.

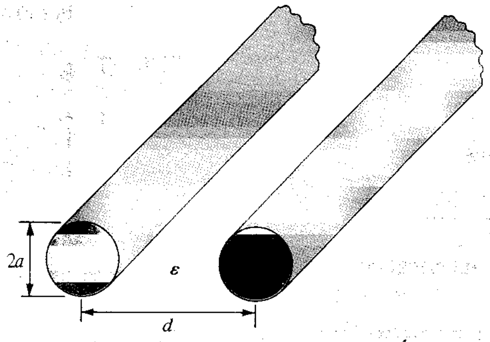

Figure 6.37 For Problem 6.40.

6.40 The capacitance per unit length of a two-wire transmission line shown in Figure 6.37 is given by

$$C = \frac{\pi \varepsilon}{\cosh^{-1}\left[\dfrac{d}{2a}\right]}$$

Determine the conductance per unit length.

***6.41** A spherical capacitor has an inner conductor of radius a carrying charge Q and maintained at zero potential. If the outer conductor contracts from a radius b to c under internal forces, prove that the work performed by the electric field as a result of the contraction is

$$W = \frac{Q^2(b - c)}{8\pi\varepsilon bc}$$

***6.42** A parallel-plate capacitor has its plates at $x = 0, d$ and the space between the plates is filled with an inhomogeneous material with permittivity $\varepsilon = \varepsilon_o\left(1 + \dfrac{x}{d}\right)$. If the plate at $x = d$ is maintained at V_o while the plate at $x = 0$ is grounded, find:

(a) V and $\mathbf{E}$

(b) $\mathbf{P}$

(c) ρ_{ps} at $x = 0, d$

6.43 A spherical capacitor has inner radius a and outer radius b and filled with an inhomogeneous dielectric with $\varepsilon = \varepsilon_o k/r^2$. Show that the capacitance of the capacitor is

$$C = \frac{4\pi\varepsilon_o k}{b - a}$$

6.44 A cylindrical capacitor with inner radius a and outer radius b is filled with an inhomogeneous dielectric having $\varepsilon = \varepsilon_o k/\rho$, where k is a constant. Calculate the capacitance per unit length of the capacitor.

6.45 If the earth is regarded a spherical capacitor, what is its capacitance? Assume the radius of the earth to be approximately 6370 km.

6.46 A point charge of 10 nC is located at point $P(0, 0, 3)$ while the conducting plane $z = 0$ is grounded. Calculate

(a) V and $\mathbf{E}$ at $R(6, 3, 5)$

(b) The force on the charge due to induced charge on the plane.

6.47 Two point charges of 3 nC and -4 nC are placed, respectively, at (0, 0, 1 m) and (0, 0, 2 m) while an infinite conducting plane is at $z = 0$. Determine

(a) The total charge induced on the plane

(b) The magnitude of the force of attraction between the charges and the plane

6.48 Two point charges of 50 nC and -20 nC are located at $(-3, 2, 4)$ and $(1, 0, 5)$ above the conducting ground plane $z = 2$. Calculate (a) the surface charge density at $(7, -2, 2)$, (b) $\mathbf{D}$ at $(3, 4, 8)$, and (c) $\mathbf{D}$ at $(1, 1, 1)$.

***6.49** A point charge of 10 μC is located at $(1, 1, 1)$, and the positive portions of the coordinate planes are occupied by three mutually perpendicular plane conductors maintained at zero potential. Find the force on the charge due to the conductors.

6.50 A point charge Q is placed between two earthed intersecting conducting planes that are inclined at 45° to each other. Determine the number of image charges and their locations.

6.51 Infinite line $x = 3, z = 4$ carries 16 nC/m and is located in free space above the conducting plane $z = 0$. (a) Find $\mathbf{E}$ at $(2, -2, 3)$. (b) Calculate the induced surface charge density on the conducting plane at $(5, -6, 0)$.

6.52 In free space, infinite planes $y = 4$ and $y = 8$ carry charges 20 nC/m^2 and 30 nC/m^2, respectively. If plane $y = 2$ is grounded, calculate $\mathbf{E}$ at $P(0, 0, 0)$ and $Q(-4, 6, 2)$.

PART 3

MAGNETOSTATICS

Chapter 7

MAGNETOSTATIC FIELDS

No honest man can be all things to all people.

—ABRAHAM LINCOLN

7.1 INTRODUCTION

In Chapters 4 to 6, we limited our discussions to static electric fields characterized by **E** or **D**. We now focus our attention on static magnetic fields, which are characterized by **H** or **B**. There are similarities and dissimilarities between electric and magnetic fields. As **E** and **D** are related according to $\mathbf{D} = \varepsilon\mathbf{E}$ for linear material space, **H** and **B** are related according to $\mathbf{B} = \mu\mathbf{H}$. Table 7.1 further shows the analogy between electric and magnetic field quantities. Some of the magnetic field quantities will be introduced later in this chapter, and others will be presented in the next. The analogy is presented here to show that most of the equations we have derived for the electric fields may be readily used to obtain corresponding equations for magnetic fields if the equivalent analogous quantities are substituted. This way it does not appear as if we are learning new concepts.

A definite link between electric and magnetic fields was established by Oersted[1] in 1820. As we have noticed, an electrostatic field is produced by static or stationary charges. If the charges are moving with constant velocity, a static magnetic (or magnetostatic) field is produced. A magnetostatic field is produced by a constant current flow (or direct current). This current flow may be due to magnetization currents as in permanent magnets, electron-beam currents as in vacuum tubes, or conduction currents as in current-carrying wires. In this chapter, we consider magnetic fields in free space due to direct current. Magnetostatic fields in material space are covered in Chapter 8.

Our study of magnetostatics is not a dispensable luxury but an indispensable necessity. The development of the motors, transformers, microphones, compasses, telephone bell ringers, television focusing controls, advertising displays, magnetically levitated high-speed vehicles, memory stores, magnetic separators, and so on, involve magnetic phenomena and play an important role in our everyday life.[2]

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[1]Hans Christian Oersted (1777–1851), a Danish professor of physics, after 13 years of frustrating efforts discovered that electricity could produce magnetism.

[2]Various applications of magnetism can be found in J. K. Watson, *Applications of Magnetism.* New York: John Wiley & Sons, 1980.

TABLE 7.1 Analogy between Electric and Magnetic Fields*

Term	Electric	Magnetic
Basic laws	$\mathbf{F} = \dfrac{Q_1 Q_2}{4\pi\varepsilon r^2}\,\mathbf{a}_r$	$d\mathbf{B} = \dfrac{\mu_o I\, d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2}$
	$\oint \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enc}}$	$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}}$
Force law	$\mathbf{F} = Q\mathbf{E}$	$\mathbf{F} = Q\mathbf{u} \times \mathbf{B}$
Source element	dQ	$Q\mathbf{u} = I\,d\mathbf{l}$
Field intensity	$E = \dfrac{V}{\ell}$ (V/m)	$H = \dfrac{I}{\ell}$ (A/m)
Flux density	$\mathbf{D} = \dfrac{\Psi}{S}$ (C/m^2)	$\mathbf{B} = \dfrac{\Psi}{S}$ (Wb/m^2)
Relationship between fields	$\mathbf{D} = \varepsilon\mathbf{E}$	$\mathbf{B} = \mu\mathbf{H}$
Potentials	$\mathbf{E} = -\nabla V$	$\mathbf{H} = -\nabla V_m \ (\mathbf{J} = 0)$
	$V = \displaystyle\int \frac{\rho_L dl}{4\pi\varepsilon r}$	$\mathbf{A} = \displaystyle\int \frac{\mu\, I\, d\mathbf{l}}{4\pi R}$
Flux	$\Psi = \int \mathbf{D} \cdot d\mathbf{S}$	$\Psi = \int \mathbf{B} \cdot d\mathbf{S}$
	$\Psi = Q = CV$	$\Psi = LI$
	$I = C\dfrac{dV}{dt}$	$V = L\dfrac{dI}{dt}$
Energy density	$w_E = \dfrac{1}{2}\mathbf{D} \cdot \mathbf{E}$	$w_m = \dfrac{1}{2}\mathbf{B} \cdot \mathbf{H}$
Poisson's equation	$\nabla^2 V = -\dfrac{\rho_v}{\varepsilon}$	$\nabla^2 \mathbf{A} = -\mu\mathbf{J}$

*A similar analogy can be found in R. S. Elliot, "Electromagnetic theory: a simplified representation," *IEEE Trans. Educ.*, vol. E-24, no. 4, Nov. 1981, pp. 294–296.

There are two major laws governing magnetostatic fields: (1) Biot–Savart's law,[3] and (2) Ampere's circuit law.[4] Like Coulomb's law, Biot–Savart's law is the general law of magnetostatics. Just as Gauss's law is a special case of Coulomb's law, Ampere's law is a special case of Biot–Savart's law and is easily applied in problems involving symmetrical current distribution. The two laws of magnetostatics are stated and applied first; their derivation is provided later in the chapter.

[3]The experiments and analyses of the effect of a current element were carried out by Ampere and by Jean-Baptiste and Felix Savart, around 1820.

[4]Andre Marie Ampere (1775–1836), a French physicist, developed Oersted's discovery and introduced the concept of current element and the force between current elements.

7.2 BIOT–SAVART'S LAW

Biot–Savart's law states that the magnetic field intensity dH produced at a point P, as shown in Figure 7.1, by the differential current element $I\,dl$ is proportional to the product $I\,dl$ and the sine of the angle α between the element and the line joining P to the element and is inversely proportional to the square of the distance R between P and the element.

That is,

$$dH \propto \frac{I\,dl \sin\alpha}{R^2} \tag{7.1}$$

or

$$dH = \frac{kI\,dl \sin\alpha}{R^2} \tag{7.2}$$

where k is the constant of proportionality. In SI units, $k = 1/4\pi$, so eq. (7.2) becomes

$$dH = \frac{I\,dl \sin\alpha}{4\pi R^2} \tag{7.3}$$

From the definition of cross product in eq. (1.21), it is easy to notice that eq. (7.3) is better put in vector form as

$$\boxed{d\mathbf{H} = \frac{I\,d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3}} \tag{7.4}$$

where $R = |\mathbf{R}|$ and $\mathbf{a}_R = \mathbf{R}/R$. Thus the direction of $d\mathbf{H}$ can be determined by the right-hand rule with the right-hand thumb pointing in the direction of the current, the right-hand fingers encircling the wire in the direction of $d\mathbf{H}$ as shown in Figure 7.2(a). Alternatively, we can use the right-handed screw rule to determine the direction of $d\mathbf{H}$: with the screw placed along the wire and pointed in the direction of current flow, the direction of advance of the screw is the direction of $d\mathbf{H}$ as in Figure 7.2(b).

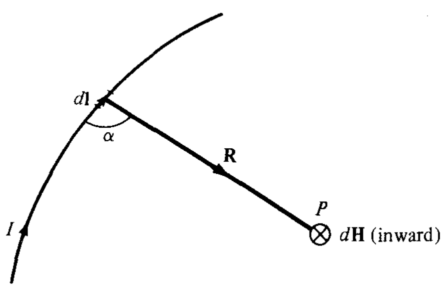


Figure 7.1 magnetic field $d\mathbf{H}$ at P due to current element $I\,d\mathbf{l}$.

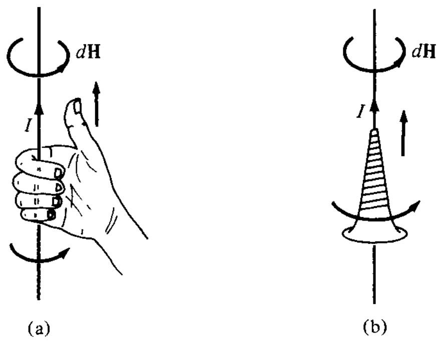


Figure 7.2 Determining the direction of $d\mathbf{H}$ using **(a)** the right-hand rule, or **(b)** the right-handed screw rule.

It is customary to represent the direction of the magnetic field intensity **H** (or current I) by a small circle with a dot or cross sign depending on whether **H** (or I) is out of, or into, the page as illustrated in Figure 7.3.

Just as we can have different charge configurations (see Figure 4.5), we can have different current distributions: line current, surface current, and volume current as shown in Figure 7.4. If we define **K** as the surface current density (in amperes/meter) and **J** as the volume current density (in amperes/meter square), the source elements are related as

$$I\,d\mathbf{l} \equiv \mathbf{K}\,dS \equiv \mathbf{J}\,dv \tag{7.5}$$

Thus in terms of the distributed current sources, the Biot–Savart law as in eq. (7.4) becomes

$$\mathbf{H} = \int_L \frac{I\,d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} \quad \text{(line current)} \tag{7.6}$$

$$\mathbf{H} = \int_S \frac{\mathbf{K}\,dS \times \mathbf{a}_R}{4\pi R^2} \quad \text{(surface current)} \tag{7.7}$$

$$\mathbf{H} = \int_v \frac{\mathbf{J}\,dv \times \mathbf{a}_R}{4\pi R^2} \quad \text{(volume current)} \tag{7.8}$$

As an example, let us apply eq. (7.6) to determine the field due to a *straight* current carrying filamentary conductor of finite length AB as in Figure 7.5. We assume that the conductor is along the z-axis with its upper and lower ends respectively subtending angles

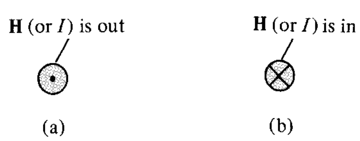


Figure 7.3 Conventional representation of **H** (or I) **(a)** out of the page and **(b)** into the page.

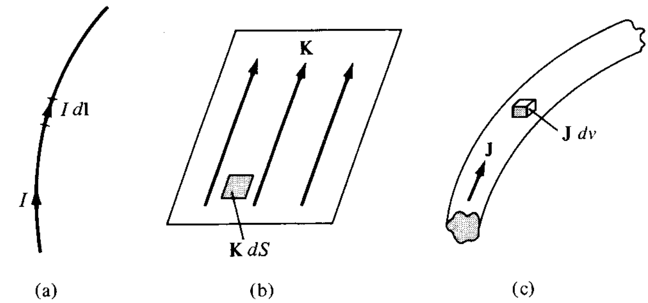


Figure 7.4 Current distributions: **(a)** line current, **(b)** surface current, **(c)** volume current.

α_2 and α_1 at P, the point at which **H** is to be determined. Particular note should be taken of this assumption as the formula to be derived will have to be applied accordingly. If we consider the contribution $d\mathbf{H}$ at P due to an element $d\mathbf{l}$ at $(0, 0, z)$,

$$d\mathbf{H} = \frac{I\, d\mathbf{l} \times \mathbf{R}}{4\pi R^3} \tag{7.9}$$

But $d\mathbf{l} = dz\, \mathbf{a}_z$ and $\mathbf{R} = \rho \mathbf{a}_\rho - z\mathbf{a}_z$, so

$$d\mathbf{l} \times \mathbf{R} = \rho\, dz\, \mathbf{a}_\phi \tag{7.10}$$

Hence,

$$\mathbf{H} = \int \frac{I\rho\, dz}{4\pi[\rho^2 + z^2]^{3/2}}\, \mathbf{a}_\phi \tag{7.11}$$

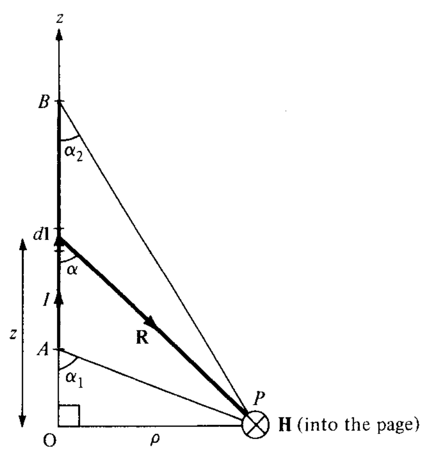


Figure 7.5 Field at point P due to a straight filamentary conductor.

Letting $z = \rho \cot \alpha$, $dz = -\rho \operatorname{cosec}^2 \alpha \, d\alpha$, and eq. (7.11) becomes

$$\mathbf{H} = -\frac{1}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{\rho^2 \operatorname{cosec}^2 \alpha \, d\alpha}{\rho^3 \operatorname{cosec}^3 \alpha} \mathbf{a}_\phi$$
$$= -\frac{I}{4\pi\rho} \mathbf{a}_\phi \int_{\alpha_1}^{\alpha_2} \sin \alpha \, d\alpha$$

or

$$\boxed{\mathbf{H} = \frac{I}{4\pi\rho} (\cos \alpha_2 - \cos \alpha_1) \mathbf{a}_\phi} \tag{7.12}$$

This expression is generally applicable for any straight filamentary conductor of finite length. Notice from eq. (7.12) that $\mathbf{H}$ is always along the unit vector $\mathbf{a}_\phi$ (i.e., along concentric circular paths) irrespective of the length of the wire or the point of interest P. As a special case, when the conductor is *semiinfinite* (with respect to P) so that point A is now at $O(0, 0, 0)$ while B is at $(0, 0, \infty)$; $\alpha_1 = 90°$, $\alpha_2 = 0°$, and eq. (7.12) becomes

$$\mathbf{H} = \frac{I}{4\pi\rho} \mathbf{a}_\phi \tag{7.13}$$

Another special case is when the conductor is *infinite* in length. For this case, point A is at $(0, 0, -\infty)$ while B is at $(0, 0, \infty)$; $\alpha_1 = 180°$, $\alpha_2 = 0°$, so eq. (7.12) reduces to

$$\boxed{\mathbf{H} = \frac{I}{2\pi\rho} \mathbf{a}_\phi} \tag{7.14}$$

To find unit vector $\mathbf{a}_\phi$ in eqs. (7.12) to (7.14) is not always easy. A simple approach is to determine $\mathbf{a}_\phi$ from

$$\boxed{\mathbf{a}_\phi = \mathbf{a}_\ell \times \mathbf{a}_\rho} \tag{7.15}$$

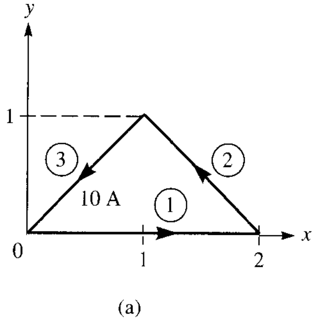
where $\mathbf{a}_\ell$ is a unit vector along the line current and $\mathbf{a}_\rho$ is a unit vector along the perpendicular line from the line current to the field point.

EXAMPLE 7.1

The conducting triangular loop in Figure 7.6(a) carries a current of 10 A. Find $\mathbf{H}$ at (0, 0, 5) due to side 1 of the loop.

Solution:

This example illustrates how eq. (7.12) is applied to any straight, thin, current-carrying conductor. The key point to keep in mind in applying eq. (7.12) is figuring out α_1, α_2, ρ, and $\mathbf{a}_\phi$. To find $\mathbf{H}$ at (0, 0, 5) due to side 1 of the loop in Figure 7.6(a), consider Figure



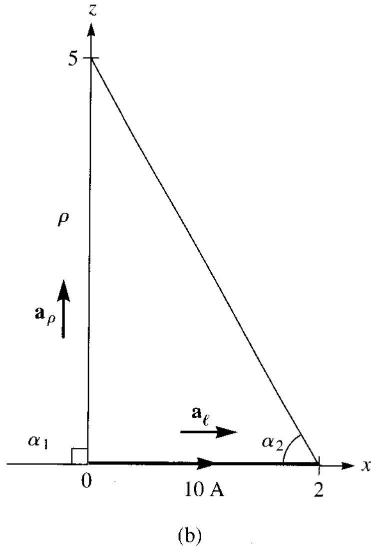


Figure 7.6 For Example 7.1: **(a)** conducting triangular loop, **(b)** side 1 of the loop.

7.6(b), where side 1 is treated as a straight conductor. Notice that we join the point of interest (0, 0, 5) to the beginning and end of the line current. Observe that α_1, α_2, and ρ are assigned in the same manner as in Figure 7.5 on which eq. (7.12) is based.

$$\cos \alpha_1 = \cos 90° = 0, \qquad \cos \alpha_2 = \frac{2}{\sqrt{29}}, \qquad \rho = 5$$

To determine $\mathbf{a}_\phi$ is often the hardest part of applying eq. (7.12). According to eq. (7.15), $\mathbf{a}_\ell = \mathbf{a}_x$ and $\mathbf{a}_\rho = \mathbf{a}_z$, so

$$\mathbf{a}_\phi = \mathbf{a}_x \times \mathbf{a}_z = -\mathbf{a}_y$$

Hence,

$$\begin{aligned}\mathbf{H}_1 &= \frac{I}{4\pi\rho}(\cos \alpha_2 - \cos \alpha_1)\mathbf{a}_\phi = \frac{10}{4\pi(5)}\left(\frac{2}{\sqrt{29}} - 0\right)(-\mathbf{a}_y) \\ &= -59.1\mathbf{a}_y \text{ mA/m}\end{aligned}$$

PRACTICE EXERCISE 7.1

Find **H** at (0, 0, 5) due to side 3 of the triangular loop in Figure 7.6(a).

Answer: $-30.63\mathbf{a}_x + 30.63\mathbf{a}_y$ mA/m.

EXAMPLE 7.2

Find **H** at $(-3, 4, 0)$ due to the current filament shown in Figure 7.7(a).

Solution:
Let $\mathbf{H} = \mathbf{H}_x + \mathbf{H}_z$, where $\mathbf{H}_x$ and $\mathbf{H}_z$ are the contributions to the magnetic field intensity at $P(-3, 4, 0)$ due to the portions of the filament along x and z, respectively.

$$\mathbf{H}_z = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\,\mathbf{a}_\phi$$

At $P(-3, 4, 0)$, $\rho = (9 + 16)^{1/2} = 5$, $\alpha_1 = 90°$, $\alpha_2 = 0°$, and $\mathbf{a}_\phi$ is obtained as a unit vector along the circular path through P on plane $z = 0$ as in Figure 7.7(b). The direction of $\mathbf{a}_\phi$ is determined using the right-handed screw rule or the right-hand rule. From the geometry in Figure 7.7(b),

$$\mathbf{a}_\phi = \sin\theta\,\mathbf{a}_x + \cos\theta\,\mathbf{a}_y = \frac{4}{5}\mathbf{a}_x + \frac{3}{5}\mathbf{a}_y$$

Alternatively, we can determine $\mathbf{a}_\phi$ from eq. (7.15). At point P, $\mathbf{a}_\ell$ and $\mathbf{a}_\rho$ are as illustrated in Figure 7.7(a) for $\mathbf{H}_z$. Hence,

$$\mathbf{a}_\phi = -\mathbf{a}_z \times \left(-\frac{3}{5}\mathbf{a}_x + \frac{4}{5}\mathbf{a}_y\right) = \frac{4}{5}\mathbf{a}_x + \frac{3}{5}\mathbf{a}_y$$

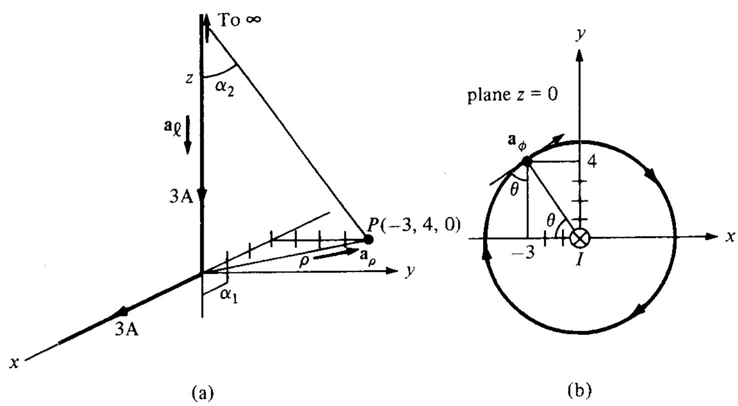


Figure 7.7 For Example 7.2: **(a)** current filament along semiinfinite x- and z-axes; $\mathbf{a}_\ell$ and $\mathbf{a}_\rho$ for $\mathbf{H}_z$ only; **(b)** determining $\mathbf{a}_\rho$ for $\mathbf{H}_z$.

as obtained before. Thus

$$\mathbf{H}_z = \frac{3}{4\pi(5)}(1 - 0)\frac{(4\mathbf{a}_x + 3\mathbf{a}_y)}{5}$$
$$= 38.2\mathbf{a}_x + 28.65\mathbf{a}_y \text{ mA/m}$$

It should be noted that in this case $\mathbf{a}_\phi$ happens to be the negative of the regular $\mathbf{a}_\phi$ of cylindrical coordinates. $\mathbf{H}_z$ could have also been obtained in cylindrical coordinates as

$$\mathbf{H}_z = \frac{3}{4\pi(5)}(1 - 0)(-\mathbf{a}_\phi)$$
$$= -47.75\mathbf{a}_\phi \text{ mA/m}$$

Similarly, for $\mathbf{H}_x$ at P, $\rho = 4$, $\alpha_2 = 0°$, $\cos\alpha_1 = 3/5$, and $\mathbf{a}_\phi = \mathbf{a}_z$ or $\mathbf{a}_\phi = \mathbf{a}_\ell \times \mathbf{a}_\rho = \mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$. Hence,

$$\mathbf{H}_x = \frac{3}{4\pi(4)}\left(1 - \frac{3}{5}\right)\mathbf{a}_z$$
$$= 23.88 \ \mathbf{a}_z \text{ mA/m}$$

Thus

$$\mathbf{H} = \mathbf{H}_x + \mathbf{H}_z = 38.2\mathbf{a}_x + 28.65\mathbf{a}_y + 23.88\mathbf{a}_z \text{ mA/m}$$

or

$$\mathbf{H} = -47.75\mathbf{a}_\phi + 23.88\mathbf{a}_z \text{ mA/m}$$

Notice that although the current filaments appear semiinfinite (they occupy the positive z- and x-axes), it is only the filament along the z-axis that is semiinfinite with respect to point P. Thus $\mathbf{H}_z$ could have been found by using eq. (7.13), but the equation could not have been used to find $\mathbf{H}_x$ because the filament along the x-axis is not semiinfinite with respect to P.

PRACTICE EXERCISE 7.2

The positive y-axis (semiinfinite line with respect to the origin) carries a filamentary current of 2 A in the $-\mathbf{a}_y$ direction. Assume it is part of a large circuit. Find $\mathbf{H}$ at

(a) $A(2, 3, 0)$

(b) $B(3, 12, -4)$

Answer: (a) $145.8\mathbf{a}_z$ mA/m, (b) $48.97\mathbf{a}_x + 36.73\mathbf{a}_z$ mA/m.

EXAMPLE 7.3

A circular loop located on $x^2 + y^2 = 9$, $z = 0$ carries a direct current of 10 A along $\mathbf{a}_\phi$. Determine $\mathbf{H}$ at (0, 0, 4) and (0, 0, −4).

Solution:

Consider the circular loop shown in Figure 7.8(a). The magnetic field intensity $d\mathbf{H}$ at point $P(0, 0, h)$ contributed by current element $I\,d\mathbf{l}$ is given by Biot–Savart's law:

$$d\mathbf{H} = \frac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

where $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$, $\mathbf{R} = (0, 0, h) - (x, y, 0) = -\rho\mathbf{a}_\rho + h\mathbf{a}_z$, and

$$d\mathbf{l} \times \mathbf{R} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & \rho\, d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = \rho h\, d\phi\, \mathbf{a}_\rho + \rho^2\, d\phi\, \mathbf{a}_z$$

Hence,

$$d\mathbf{H} = \frac{I}{4\pi[\rho^2 + h^2]^{3/2}}(\rho h\, d\phi\, \mathbf{a}_\rho + \rho^2\, d\phi\, \mathbf{a}_z) = dH_\rho\, \mathbf{a}_\rho + dH_z\, \mathbf{a}_z$$

By symmetry, the contributions along $\mathbf{a}_\rho$ add up to zero because the radial components produced by pairs of current element 180° apart cancel. This may also be shown mathematically by writing $\mathbf{a}_\rho$ in rectangular coordinate systems (i.e., $\mathbf{a}_\rho = \cos\phi\, \mathbf{a}_x + \sin\phi\, \mathbf{a}_y$).

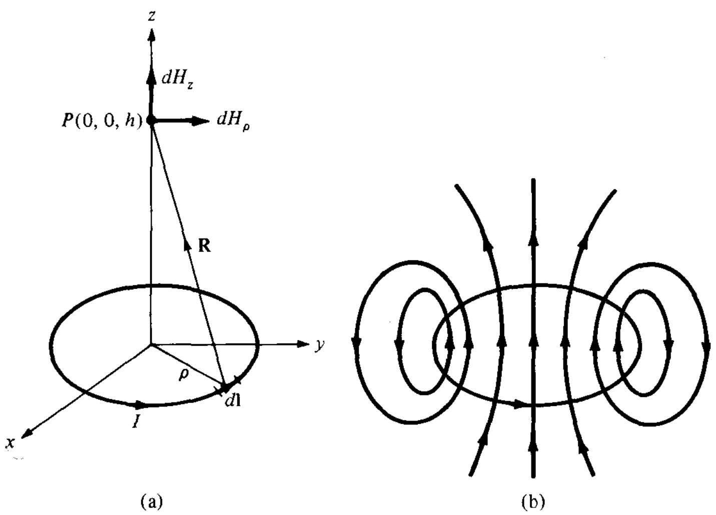


Figure 7.8 For Example 7.3: **(a)** circular current loop, **(b)** flux lines due to the current loop.

Integrating $\cos\phi$ or $\sin\phi$ over $0 \le \phi \le 2\pi$ gives zero, thereby showing that $\mathbf{H}_\rho = 0$. Thus

$$\mathbf{H} = \int dH_z\,\mathbf{a}_z = \int_0^{2\pi} \frac{I\rho^2\,d\phi\,\mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}} = \frac{I\rho^2 2\pi\mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}}$$

or

$$\mathbf{H} = \frac{I\rho^2\mathbf{a}_z}{2[\rho^2 + h^2]^{3/2}}$$

(a) Substituting $I = 10\,A$, $\rho = 3$, $h = 4$ gives

$$\mathbf{H}(0, 0, 4) = \frac{10\,(3)^2\,\mathbf{a}_z}{2[9 + 16]^{3/2}} = 0.36\mathbf{a}_z \text{ A/m}$$

(b) Notice from $d\mathbf{l} \times \mathbf{R}$ above that if h is replaced by $-h$, the z-component of $d\mathbf{H}$ remains the same while the ρ-component still adds up to zero due to the axial symmetry of the loop. Hence

$$\mathbf{H}(0, 0, -4) = \mathbf{H}(0, 0, 4) = 0.36\mathbf{a}_z \text{ A/m}$$

The flux lines due to the circular current loop are sketched in Figure 7.8(b).

PRACTICE EXERCISE 7.3

A thin ring of radius 5 cm is placed on plane $z = 1$ cm so that its center is at (0, 0, 1 cm). If the ring carries 50 mA along $\mathbf{a}_\phi$, find $\mathbf{H}$ at

(a) (0, 0, −1 cm)

(b) (0, 0, 10 cm)

Answer: (a) $400\mathbf{a}_z$ mA/m, (b) $57.3\mathbf{a}_z$ mA/m.

EXAMPLE 7.4

A solenoid of length ℓ and radius a consists of N turns of wire carrying current I. Show that at point P along its axis,

$$\mathbf{H} = \frac{nI}{2}(\cos\theta_2 - \cos\theta_1)\mathbf{a}_z$$

where $n = N/\ell$, θ_1 and θ_2 are the angles subtended at P by the end turns as illustrated in Figure 7.9. Also show that if $\ell \gg a$, at the center of the solenoid,

$$\mathbf{H} = nI\mathbf{a}_z$$

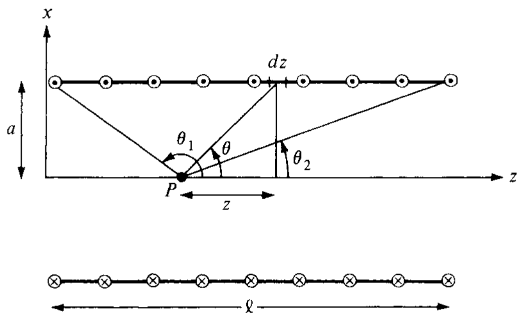


Figure 7.9 For Example 7.4; cross section of a solenoid.

Solution:

Consider the cross section of the solenoid as shown in Figure 7.9. Since the solenoid consists of circular loops, we apply the result of Example 7.3. The contribution to the magnetic field H at P by an element of the solenoid of length dz is

$$dH_z = \frac{I\,dl\,a^2}{2[a^2 + z^2]^{3/2}} = \frac{Ia^2 n\,dz}{2[a^2 + z^2]^{3/2}}$$

where $dl = n\,dz = (N/\ell)\,dz$. From Figure 7.9, $\tan\theta = a/z$; that is,

$$dz = -a\,\text{cosec}^2\,\theta\,d\theta = -\frac{[z^2 + a^2]^{3/2}}{a^2}\sin\theta\,d\theta$$

Hence,

$$dH_z = -\frac{nI}{2}\sin\theta\,d\theta$$

or

$$H_z = -\frac{nI}{2}\int_{\theta_1}^{\theta_2}\sin\theta\,d\theta$$

Thus

$$\mathbf{H} = \frac{nI}{2}(\cos\theta_2 - \cos\theta_1)\,\mathbf{a}_z$$

as required. Substituting $n = N/\ell$ gives

$$\mathbf{H} = \frac{NI}{2\ell}(\cos\theta_2 - \cos\theta_1)\,\mathbf{a}_z$$

At the center of the solenoid,

$$\cos\theta_2 = \frac{\ell/2}{[a^2 + \ell^2/4]^{1/2}} = -\cos\theta_1$$

and

$$\mathbf{H} = \frac{In\ell}{2[a^2 + \ell^2/4]^{1/2}}\,\mathbf{a}_z$$

If $\ell \gg a$ or $\theta_2 \simeq 0°$, $\theta_1 \simeq 180°$,

$$\mathbf{H} = nI\mathbf{a}_z = \frac{NI}{\ell}\,\mathbf{a}_z$$

PRACTICE EXERCISE 7.4

If the solenoid of Figure 7.9 has 2,000 turns, a length of 75 cm, a radius of 5 cm, and carries a current of 50 mA along $\mathbf{a}_\phi$, find $\mathbf{H}$ at

(a) (0, 0, 0)

(b) (0, 0, 75 cm)

(c) (0, 0, 50 cm)

Answer: (a) $66.52\mathbf{a}_z$ A/m, (b) $66.52\mathbf{a}_z$ A/m, (c) $131.7\mathbf{a}_z$ A/m.

7.3 AMPERE'S CIRCUIT LAW—MAXWELL'S EQUATION

Ampere's circuit law states that the line integral of the tangential component of **H** around a *closed* path is the same as the net current I_{enc} enclosed by the path.

In other words, the circulation of **H** equals I_{enc}; that is,

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc}} \tag{7.16}$$

Ampere's law is similar to Gauss's law and it is easily applied to determine **H** when the current distribution is symmetrical. It should be noted that eq. (7.16) always holds whether the current distribution is symmetrical or not but we can only use the equation to determine **H** when symmetrical current distribution exists. Ampere's law is a special case of Biot–Savart's law; the former may be derived from the latter.

By applying Stoke's theorem to the left-hand side of eq. (7.16), we obtain

$$I_{enc} = \oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} \tag{7.17}$$

But

$$I_{\text{enc}} = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{7.18}$$

Comparing the surface integrals in eqs. (7.17) and (7.18) clearly reveals that

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}} \tag{7.19}$$

This is the third Maxwell's equation to be derived; it is essentially Ampere's law in differential (or point) form whereas eq. (7.16) is the integral form. From eq. (7.19), we should observe that $\nabla \times \mathbf{H} = \mathbf{J} \neq 0$; that is, magnetostatic field is not conservative.

7.4 APPLICATIONS OF AMPERE'S LAW

We now apply Ampere's circuit law to determine **H** for some symmetrical current distributions as we did for Gauss's law. We will consider an infinite line current, an infinite current sheet, and an infinitely long coaxial transmission line.

A. Infinite Line Current

Consider an infinitely long filamentary current I along the z-axis as in Figure 7.10. To determine **H** at an observation point P, we allow a closed path pass through P. This path, on which Ampere's law is to be applied, is known as an *Amperian path* (analogous to the term Gaussian surface). We choose a concentric circle as the Amperian path in view of eq. (7.14), which shows that **H** is constant provided ρ is constant. Since this path encloses the whole current I, according to Ampere's law

$$I = \int H_\phi \mathbf{a}_\phi \cdot \rho \, d\phi \, \mathbf{a}_\phi = H_\phi \int \rho \, d\phi = H_\phi \cdot 2\pi\rho$$

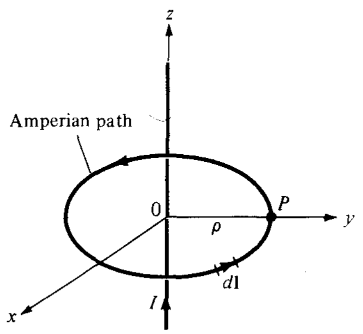


Figure 7.10 Ampere's law applied to an infinite filamentary line current.

or

$$\boxed{\mathbf{H} = \frac{I}{2\pi\rho}\,\mathbf{a}_\phi} \tag{7.20}$$

as expected from eq. (7.14).

B. Infinite Sheet of Current

Consider an infinite current sheet in the $z = 0$ plane. If the sheet has a uniform current density $\mathbf{K} = K_y\mathbf{a}_y$ A/m as shown in Figure 7.11, applying Ampere's law to the rectangular closed path (Amperian path) gives

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc} = K_y b \tag{7.21a}$$

To evaluate the integral, we first need to have an idea of what $\mathbf{H}$ is like. To achieve this, we regard the infinite sheet as comprising of filaments; $d\mathbf{H}$ above or below the sheet due to a pair of filamentary currents can be found using eqs. (7.14) and (7.15). As evident in Figure 7.11(b), the resultant $d\mathbf{H}$ has only an x-component. Also, $\mathbf{H}$ on one side of the sheet is the negative of that on the other side. Due to the infinite extent of the sheet, the sheet can be regarded as consisting of such filamentary pairs so that the characteristics of $\mathbf{H}$ for a pair are the same for the infinite current sheets, that is,

$$\mathbf{H} = \begin{cases} H_o\mathbf{a}_x & z > 0 \\ -H_o\mathbf{a}_x & z < 0 \end{cases} \tag{7.21b}$$

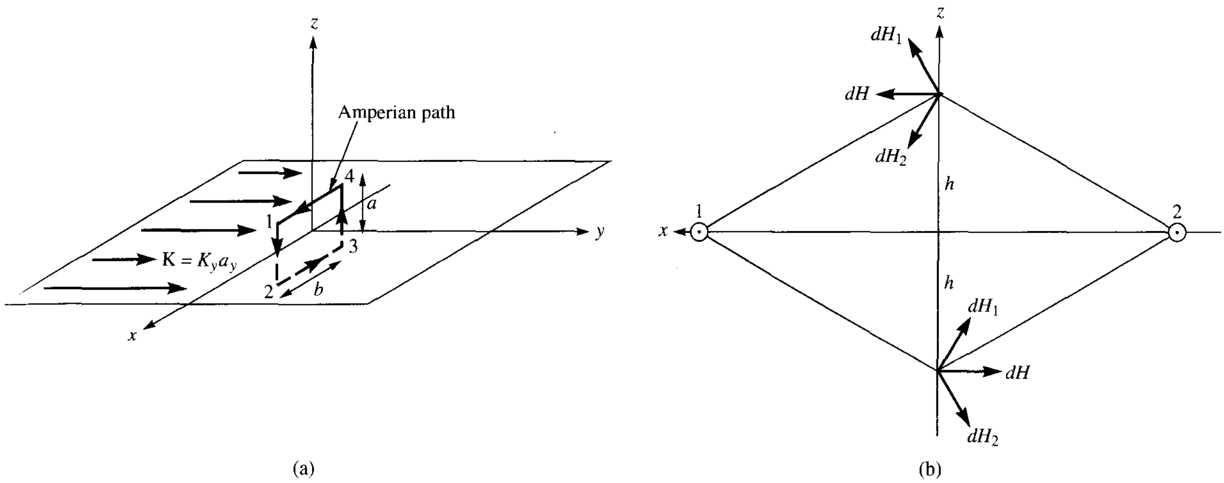


Figure 7.11 Application of Ampere's law to an infinite sheet: **(a)** closed path 1-2-3-4-1, **(b)** symmetrical pair of current filaments with current along $\mathbf{a}_y$.

where H_o is yet to be determined. Evaluating the line integral of $\mathbf{H}$ in eq. (7.21b) along the closed path in Figure 7.11(a) gives

$$\begin{aligned}\oint \mathbf{H} \cdot d\mathbf{l} &= \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1\right) \mathbf{H} \cdot d\mathbf{l} \\ &= 0(-a) + (-H_o)(-b) + 0(a) + H_o(b) \\ &= 2H_o b\end{aligned} \tag{7.21c}$$

From eqs. (7.21a) and (7.21c), we obtain $H_o = \frac{1}{2} K_y$. Substituting H_o in eq. (7.21b) gives

$$\mathbf{H} = \begin{cases} \frac{1}{2} K_y \mathbf{a}_x, & z > 0 \\ -\frac{1}{2} K_y \mathbf{a}_x, & z < 0 \end{cases} \tag{7.22}$$

In general, for an infinite sheet of current density $\mathbf{K}$ A/m,

$$\boxed{\mathbf{H} = \frac{1}{2} \mathbf{K} \times \mathbf{a}_n} \tag{7.23}$$

where $\mathbf{a}_n$ is a unit normal vector directed from the current sheet to the point of interest.

C. Infinitely Long Coaxial Transmission Line

Consider an infinitely long transmission line consisting of two concentric cylinders having their axes along the z-axis. The cross section of the line is shown in Figure 7.12, where the z-axis is out of the page. The inner conductor has radius a and carries current I while the outer conductor has inner radius b and thickness t and carries return current $-I$. We want to determine $\mathbf{H}$ everywhere assuming that current is uniformly distributed in both conductors. Since the current distribution is symmetrical, we apply Ampere's law along the Am-

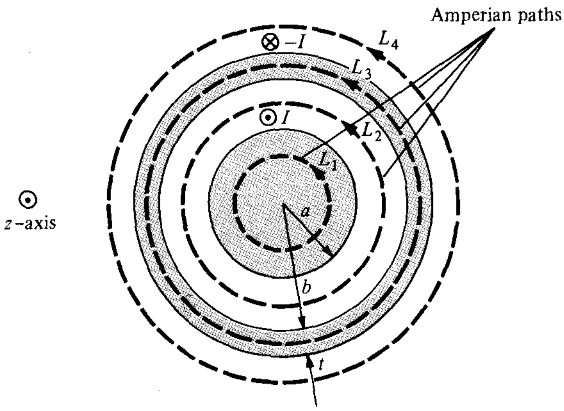


Figure 7.12 Cross section of the transmission line; the positive z-direction is out of the page.

perian path for each of the four possible regions: $0 \leq \rho \leq a$, $a \leq \rho \leq b$, $b \leq \rho \leq b + t$, and $\rho \geq b + t$.

For region $0 \leq \rho \leq a$, we apply Ampere's law to path L_1, giving

$$\oint_{L_1} \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{S} \tag{7.24}$$

Since the current is uniformly distributed over the cross section,

$$\mathbf{J} = \frac{I}{\pi a^2} \mathbf{a}_z, \qquad d\mathbf{S} = \rho \, d\phi \, d\rho \, \mathbf{a}_z$$

$$I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{S} = \frac{I}{\pi a^2} \iint \rho \, d\phi \, d\rho = \frac{I}{\pi a^2} \pi \rho^2 = \frac{I\rho^2}{a^2}$$

Hence eq. (7.24) becomes

$$H_\phi \int dl = H_\phi \, 2\pi\rho = \frac{I\rho^2}{a^2}$$

or

$$H_\phi = \frac{I\rho}{2\pi a^2} \tag{7.25}$$

For region $a \leq \rho \leq b$, we use path L_2 as the Amperian path,

$$\oint_{L_2} \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}} = I$$

$$H_\phi 2\pi\rho = I$$

or

$$H_\phi = \frac{I}{2\pi\rho} \tag{7.26}$$

since the whole current I is enclosed by L_2. Notice that eq. (7.26) is the same as eq. (7.14) and it is independent of a. For region $b \leq \rho \leq b + t$, we use path L_3, getting

$$\oint \mathbf{H} \cdot d\mathbf{l} = H_\phi \cdot 2\pi\phi = I_{\text{enc}} \tag{7.27a}$$

where

$$I_{\text{enc}} = I + \int \mathbf{J} \cdot d\mathbf{S}$$

and $\mathbf{J}$ in this case is the current density (current per unit area) of the outer conductor and is along $-\mathbf{a}_z$, that is,

$$\mathbf{J} = -\frac{I}{\pi[(b + t)^2 - b^2]}\,\mathbf{a}_z$$

Thus

$$\begin{aligned} I_{\text{enc}} &= I - \frac{I}{\pi[(b + t)^2 - t^2]} \int_{\phi=0}^{2\pi} \int_{\rho=b}^{\rho} \rho\, d\rho\, d\phi \\ &= I\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right] \end{aligned}$$

Substituting this in eq. (7.27a), we have

$$H_\phi = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right] \tag{7.27b}$$

For region $\rho \geq b + t$, we use path L_4, getting

$$\oint_{L_4} \mathbf{H} \cdot d\mathbf{l} = I - I = 0$$

or

$$H_\phi = 0 \tag{7.28}$$

Putting eqs. (7.25) to (7.28) together gives

$$\mathbf{H} = \begin{cases} \dfrac{I\rho}{2\pi a^2}\,\mathbf{a}_\phi, & 0 \leq \rho \leq a \\ \dfrac{I}{2\pi\rho}\,\mathbf{a}_\phi, & a \leq \rho \leq b \\ \dfrac{I}{2\pi\rho}\left[1 - \dfrac{\rho^2 - b^2}{t^2 + 2bt}\right]\mathbf{a}_\phi, & b \leq \rho \leq b + t \\ 0, & \rho \geq b + t \end{cases} \tag{7.29}$$

The magnitude of $\mathbf{H}$ is sketched in Figure 7.13.

Notice from these examples that the ability to take $\mathbf{H}$ from under the integral sign is the key to using Ampere's law to determine $\mathbf{H}$. In other words, Ampere's law can only be used to find $\mathbf{H}$ due to symmetric current distributions for which it is possible to find a closed path over which $\mathbf{H}$ is constant in magnitude.

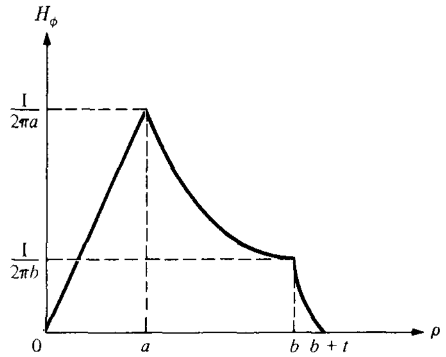


Figure 7.13 Plot of H_ϕ against ρ.

EXAMPLE 7.5

Planes $z = 0$ and $z = 4$ carry current $\mathbf{K} = -10\mathbf{a}_x$ A/m and $\mathbf{K} = 10\mathbf{a}_x$ A/m, respectively. Determine $\mathbf{H}$ at

(a) (1, 1, 1)

(b) (0, −3, 10)

Solution:

Let the parallel current sheets be as in Figure 7.14. Also let

$$\mathbf{H} = \mathbf{H}_o + \mathbf{H}_4$$

where $\mathbf{H}_o$ and $\mathbf{H}_4$ are the contributions due to the current sheets $z = 0$ and $z = 4$, respectively. We make use of eq. (7.23).

(a) At (1, 1, 1), which is between the plates ($0 < z = 1 < 4$),

$$\mathbf{H}_o = 1/2\, \mathbf{K} \times \mathbf{a}_n = 1/2\,(-10\mathbf{a}_x) \times \mathbf{a}_z = 5\mathbf{a}_y \text{ A/m}$$

$$\mathbf{H}_4 = 1/2\, \mathbf{K} \times \mathbf{a}_n = 1/2\,(10\mathbf{a}_x) \times (-\mathbf{a}_z) = 5\mathbf{a}_y \text{ A/m}$$

Hence,

$$\mathbf{H} = 10\mathbf{a}_y \text{ A/m}$$

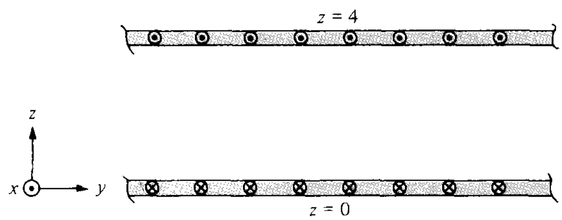


Figure 7.14 For Example 7.5; parallel infinite current sheets.

(b) At (0, −3, 10), which is above the two sheets ($z = 10 > 4 > 0$),

$$\mathbf{H}_o = 1/2\,(-10\mathbf{a}_x) \times \mathbf{a}_z = 5\mathbf{a}_y \text{ A/m}$$
$$\mathbf{H}_4 = 1/2\,(10\mathbf{a}_x) \times \mathbf{a}_z = -5\mathbf{a}_y \text{ A/m}$$

Hence,

$$\mathbf{H} = 0 \text{ A/m}$$

PRACTICE EXERCISE 7.5

Plane $y = 1$ carries current $\mathbf{K} = 50\mathbf{a}_z$ mA/m. Find $\mathbf{H}$ at

(a) (0, 0, 0)

(b) (1, 5, −3)

Answer: (a) $25\mathbf{a}_x$ mA/m, (b) $-25\mathbf{a}_x$ mA/m.

EXAMPLE 7.6

A toroid whose dimensions are shown in Figure 7.15 has N turns and carries current I. De termine H inside and outside the toroid.

Solution:

We apply Ampere's circuit law to the Amperian path, which is a circle of radius ρ show dotted in Figure 7.15. Since N wires cut through this path each carrying current I, the ne current enclosed by the Amperian path is NI. Hence,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc} \rightarrow H \cdot 2\pi\rho = NI$$

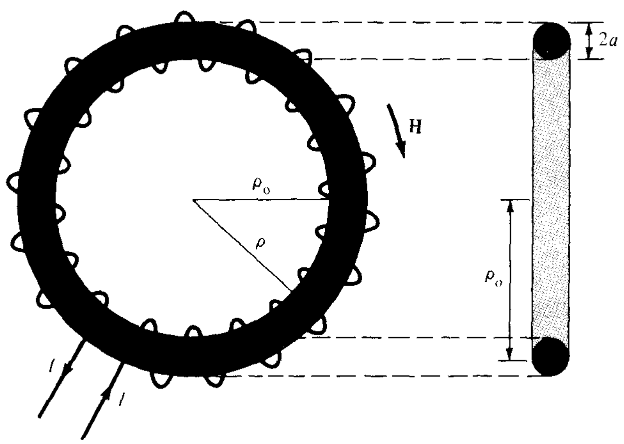


Figure 7.15 For Example 7.6; a toroid with a circular cross section.

or

$$H = \frac{NI}{2\pi\rho}, \qquad \text{for } \rho_o - a < \rho < \rho_o + a$$

where ρ_o is the mean radius of the toroid as shown in Figure 7.15. An approximate value of H is

$$H_{\text{approx}} = \frac{NI}{2\pi\rho_o} = \frac{NI}{\ell}$$

Notice that this is the same as the formula obtained for H for points well inside a very long solenoid ($\ell \gg a$). Thus a straight solenoid may be regarded as a special toroidal coil for which $\rho_o \to \infty$. Outside the toroid, the current enclosed by an Amperian path is $NI - NI = 0$ and hence $H = 0$.

PRACTICE EXERCISE 7.6

A toroid of circular cross section whose center is at the origin and axis the same as the z-axis has 1000 turns with $\rho_o = 10$ cm, $a = 1$ cm. If the toroid carries a 100-mA current, find $|H|$ at

(a) (3 cm, −4 cm, 0)

(b) (6 cm, 9 cm, 0)

Answer: (a) 0, (b) 147.1 A/m.

7.5 MAGNETIC FLUX DENSITY—MAXWELL'S EQUATION

The magnetic flux density $\mathbf{B}$ is similar to the electric flux density $\mathbf{D}$. As $\mathbf{D} = \varepsilon_o\mathbf{E}$ in free space, the magnetic flux density $\mathbf{B}$ is related to the magnetic field intensity $\mathbf{H}$ according to

$$\boxed{\mathbf{B} = \mu_o\mathbf{H}} \tag{7.30}$$

where μ_o is a constant known as the *permeability of free space*. The constant is in henrys/meter (H/m) and has the value of

$$\boxed{\mu_o = 4\pi \times 10^{-7} \text{ H/m}} \tag{7.31}$$

The precise definition of the magnetic field $\mathbf{B}$, in terms of the magnetic force, will be given in the next chapter.

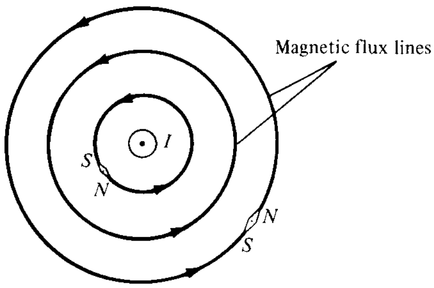


Figure 7.16 Magnetic flux lines due to a straight wire with current coming out of the page.

The magnetic flux through a surface S is given by

$$\boxed{\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}} \tag{7.32}$$

where the magnetic flux Ψ is in webers (Wb) and the magnetic flux density is in webers/square meter (Wb/m^2) or teslas.

The magnetic flux line is the path to which $\mathbf{B}$ is tangential at every point in a magnetic field. It is the line along which the needle of a magnetic compass will orient itself if placed in the magnetic field. For example, the magnetic flux lines due to a straight long wire are shown in Figure 7.16. The flux lines are determined using the same principle followed in Section 4.10 for the electric flux lines. The direction of $\mathbf{B}$ is taken as that indicated as "north" by the needle of the magnetic compass. Notice that each flux line is closed and has no beginning or end. Though Figure 7.16 is for a straight, current-carrying conductor, it is generally true that magnetic flux lines are closed and do not cross each other regardless of the current distribution.

In an electrostatic field, the flux passing through a closed surface is the same as the charge enclosed; that is, $\Psi = \oint \mathbf{D} \cdot d\mathbf{S} = Q$. Thus it is possible to have an isolated electric charge as shown in Figure 7.17(a), which also reveals that electric flux lines are not necessarily closed. Unlike electric flux lines, magnetic flux lines always close upon themselves as in Figure 7.17(b). This is due to the fact that *it is not possible to have isolated magnetic*

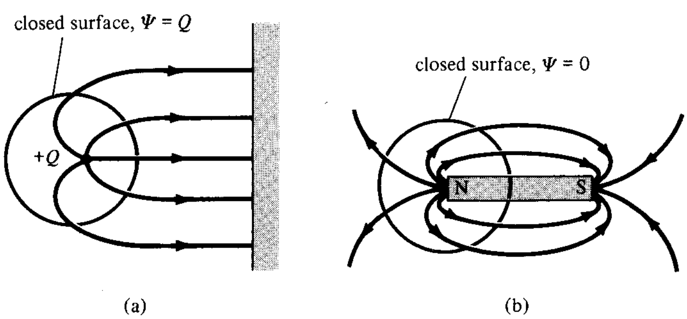


Figure 7.17 Flux leaving a closed surface due to: **(a)** isolated electric charge $\Psi = \oint_S \mathbf{D} \cdot d\mathbf{S} = Q$, **(b)** magnetic charge, $\Psi = \oint_S \mathbf{B} \cdot d\mathbf{S} = 0$.

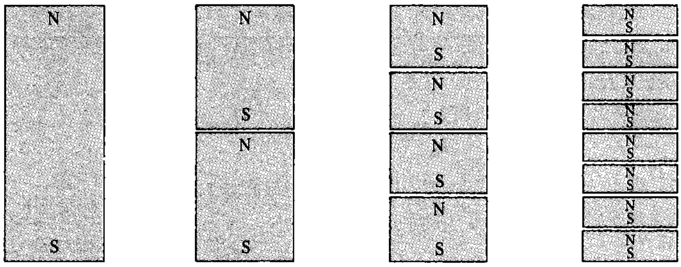


Figure 7.18 Successive division of a bar magnet results in pieces with north and south poles, showing that magnetic poles cannot be isolated.

poles (or magnetic charges). For example, if we desire to have an isolated magnetic pole by dividing a magnetic bar successively into two, we end up with pieces each having north and south poles as illustrated in Figure 7.18. We find it impossible to separate the north pole from the south pole.

An **isolated magnetic** charge does not exist.

Thus the total flux through a closed surface in a magnetic field must be zero; that is,

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{S} = 0} \tag{7.33}$$

This equation is referred to as the *law of conservation of magnetic flux* or *Gauss's law for magnetostatic fields* just as $\oint \mathbf{D} \cdot d\mathbf{S} = Q$ is Gauss's law for electrostatic fields. Although the magnetostatic field is not conservative, magnetic flux is conserved.

By applying the divergence theorem to eq. (7.33), we obtain

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{B}\, dv = 0$$

or

$$\boxed{\nabla \cdot \mathbf{B} = 0} \tag{7.34}$$

This equation is the fourth Maxwell's equation to be derived. Equation (7.33) or (7.34) shows that magnetostatic fields have no sources or sinks. Equation (7.34) suggests that magnetic field lines are always continuous.

7.6 MAXWELL'S EQUATIONS FOR STATIC EM FIELDS

Having derived Maxwell's four equations for static electromagnetic fields, we may take a moment to put them together as in Table 7.2. From the table, we notice that the order in which the equations were derived has been changed for the sake of clarity.

TABLE 7.2 Maxwell's Equations for Static EM Fields

Differential (or Point) Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of magnetic monopole
$\nabla \times \mathbf{E} = 0$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = 0$	Conservativeness of electrostatic field
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$	Ampere's law

The choice between differential and integral forms of the equations depends on a given problem. It is evident from Table 7.2 that a vector field is defined completely by specifying its curl and divergence. A field can only be electric or magnetic if it satisfies the corresponding Maxwell's equations (see Problems 7.26 and 7.27). It should be noted that Maxwell's equations as in Table 7.2 are only for static EM fields. As will be discussed in Chapter 9, the divergence equations will remain the same for time-varying EM fields but the curl equations will have to be modified.

7.7 MAGNETIC SCALAR AND VECTOR POTENTIALS

We recall that some electrostatic field problems were simplified by relating the electric potential V to the electric field intensity $\mathbf{E}$ ($\mathbf{E} = -\nabla V$). Similarly, we can define a potential associated with magnetostatic field $\mathbf{B}$. In fact, the magnetic potential could be scalar V_m or vector $\mathbf{A}$. To define V_m and $\mathbf{A}$ involves recalling two important identities (see Example 3.9 and Practice Exercise 3.9):

$$\nabla \times (\nabla V) = 0 \tag{7.35a}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{7.35b}$$

which must always hold for any scalar field V and vector field $\mathbf{A}$.

Just as $\mathbf{E} = -\nabla V$, we define the *magnetic scalar potential* V_m (in amperes) as related to $\mathbf{H}$ according to

$$\boxed{\mathbf{H} = -\nabla V_m} \qquad \text{if } \mathbf{J} = 0 \tag{7.36}$$

The condition attached to this equation is important and will be explained. Combining eq. (7.36) and eq. (7.19) gives

$$\mathbf{J} = \nabla \times \mathbf{H} = \nabla \times (-\nabla V_m) = 0 \tag{7.37}$$

since V_m must satisfy the condition in eq. (7.35a). Thus the magnetic scalar potential V_m is only defined in a region where $\mathbf{J} = 0$ as in eq. (7.36). We should also note that V_m satisfies Laplace's equation just as V does for electrostatic fields; hence,

$$\nabla^2 V_m = 0, \qquad (\mathbf{J} = 0) \tag{7.38}$$

We know that for a magnetostatic field, $\nabla \cdot \mathbf{B} = 0$ as stated in eq. (7.34). In order to satisfy eqs. (7.34) and (7.35b) simultaneously, we can define the *vector magnetic potential* $\mathbf{A}$ (in Wb/m) such that

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \tag{7.39}$$

Just as we defined

$$V = \int \frac{dQ}{4\pi\varepsilon_o r} \tag{7.40}$$

we can define

$$\boxed{\mathbf{A} = \int_L \frac{\mu_o I \, d\mathbf{l}}{4\pi R}} \quad \text{for line current} \tag{7.41}$$

$$\boxed{\mathbf{A} = \int_S \frac{\mu_o \mathbf{K} \, dS}{4\pi R}} \quad \text{for surface current} \tag{7.42}$$

$$\boxed{\mathbf{A} = \int_v \frac{\mu_o \mathbf{J} \, dv}{4\pi R}} \quad \text{for volume current} \tag{7.43}$$

Rather than obtaining eqs. (7.41) to (7.43) from eq. (7.40), an alternative approach would be to obtain eqs. (7.41) to (7.43) from eqs. (7.6) to (7.8). For example, we can derive eq. (7.41) from eq. (7.6) in conjunction with eq. (7.39). To do this, we write eq. (7.6) as

$$\mathbf{B} = \frac{\mu_o}{4\pi} \int_L \frac{I \, d\mathbf{l}' \times \mathbf{R}}{R^3} \tag{7.44}$$

where $\mathbf{R}$ is the distance vector from the line element $d\mathbf{l}'$ at the source point (x', y', z') to the field point (x, y, z) as shown in Figure 7.19 and $R = |\mathbf{R}|$, that is,

$$R = |\mathbf{r} - \mathbf{r}'| = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2} \tag{7.45}$$

Hence,

$$\nabla\left(\frac{1}{R}\right) = -\frac{(x - x')\mathbf{a}_x + (y - y')\mathbf{a}_y + (z - z')\mathbf{a}_z}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} = -\frac{\mathbf{R}}{R^3}$$

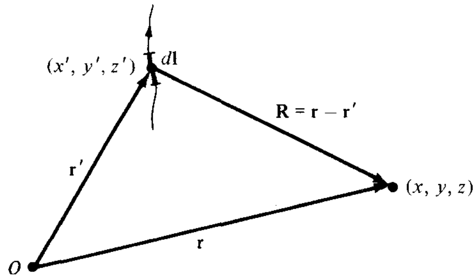


Figure 7.19 Illustration of the source point (x', y', z') and the field point (x, y, z).

or

$$\frac{\mathbf{R}}{R^3} = -\nabla\left(\frac{1}{R}\right) \quad \left(= \frac{\mathbf{a}_R}{R^2}\right) \tag{7.46}$$

where the differentiation is with respect to x, y, and z. Substituting this into eq. (7.44), we obtain

$$\mathbf{B} = -\frac{\mu_o}{4\pi} \int_L I\, d\mathbf{l}' \times \nabla\left(\frac{1}{R}\right) \tag{7.47}$$

We apply the vector identity

$$\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + (\nabla f) \times \mathbf{F} \tag{7.48}$$

where f is a scalar field and $\mathbf{F}$ is a vector field. Taking $f = 1/R$ and $\mathbf{F} = d\mathbf{l}'$, we have

$$d\mathbf{l}' \times \nabla\left(\frac{1}{R}\right) = \frac{1}{R}\nabla \times d\mathbf{l}' - \nabla \times \left(\frac{d\mathbf{l}'}{R}\right)$$

Since ∇ operates with respect to (x, y, z) while $d\mathbf{l}'$ is a function of (x', y', z'), $\nabla \times d\mathbf{l}' = 0$. Hence,

$$d\mathbf{l}' \times \nabla\left(\frac{1}{R}\right) = -\nabla \times \frac{d\mathbf{l}'}{R} \tag{7.49}$$

With this equation, eq. (7.47) reduces to

$$\mathbf{B} = \nabla \times \int_L \frac{\mu_o I\, d\mathbf{l}'}{4\pi R} \tag{7.50}$$

Comparing eq. (7.50) with eq. (7.39) shows that

$$\mathbf{A} = \int_L \frac{\mu_o I\, d\mathbf{l}'}{4\pi R}$$

verifying eq. (7.41).

By substituting eq. (7.39) into eq. (7.32) and applying Stokes's theorem, we obtain

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_L \mathbf{A} \cdot d\mathbf{l}$$

or

$$\boxed{\Psi = \oint_L \mathbf{A} \cdot d\mathbf{l}} \tag{7.51}$$

Thus the magnetic flux through a given area can be found using either eq. (7.32) or (7.51). Also, the magnetic field can be determined using either V_m or $\mathbf{A}$; the choice is dictated by the nature of the given problem except that V_m can only be used in a source-free region. The use of the magnetic vector potential provides a powerful, elegant approach to solving EM problems, particularly those relating to antennas. As we shall notice in Chapter 13, it is more convenient to find $\mathbf{B}$ by first finding $\mathbf{A}$ in antenna problems.

EXAMPLE 7.7

Given the magnetic vector potential $\mathbf{A} = -\rho^2/4\ \mathbf{a}_z$ Wb/m, calculate the total magnetic flux crossing the surface $\phi = \pi/2$, $1 \leq \rho \leq 2$ m, $0 \leq z \leq 5$ m.

Solution:

We can solve this problem in two different ways: using eq. (7.32) or eq. (7.51).

Method 1:

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial \rho}\mathbf{a}_\phi = \frac{\rho}{2}\mathbf{a}_\phi, \qquad d\mathbf{S} = d\rho\, dz\, \mathbf{a}_\phi$$

Hence,

$$\Psi = \int \mathrm{B} \cdot d\mathbf{S} = \frac{1}{2}\int_{z=0}^{5}\int_{\rho=1}^{2} \rho\, d\rho\, dz = \frac{1}{4}\rho^2 \Big|_2^1 (5) = \frac{15}{4}$$

$$\Psi = 3.75 \text{ Wb}$$

Method 2:
We use

$$\Psi = \oint_L \mathbf{A} \cdot d\mathbf{l} = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4$$

where L is the path bounding surface S; Ψ_1, Ψ_2, Ψ_3, and Ψ_4 are, respectively, the evaluations of $\int \mathbf{A} \cdot d\mathbf{l}$ along the segments of L labeled 1 to 4 in Figure 7.20. Since $\mathbf{A}$ has only a z-component,

$$\Psi_1 = 0 = \Psi_3$$

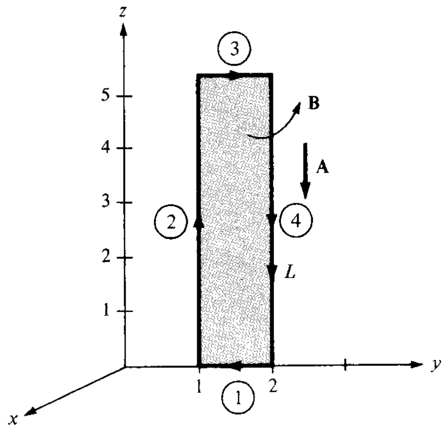


Figure 7.20 For Example 7.7.

That is,

$$\begin{aligned}\Psi = \Psi_2 + \Psi_4 &= -\frac{1}{4}\left[(1)^2 \int_0^5 dz + (2)^2 \int_5^0 dz\right] \\ &= -\frac{1}{4}(1-4)(5) = \frac{15}{4} \\ &= 3.75 \text{ Wb}\end{aligned}$$

as obtained previously. Note that the direction of the path L must agree with that of $d\mathbf{S}$.

PRACTICE EXERCISE 7.7

A current distribution gives rise to the vector magnetic potential $\mathbf{A} = x^2y\mathbf{a}_x + y^2x\mathbf{a}_y - 4xyz\mathbf{a}_z$ Wb/m. Calculate

(a) $\mathbf{B}$ at $(-1, 2, 5)$

(b) The flux through the surface defined by $z = 1, 0 \leq x \leq 1, -1 \leq y \leq 4$

Answer: (a) $20\mathbf{a}_x + 40\mathbf{a}_y + 3\mathbf{a}_z$ Wb/m^2, (b) 20 Wb.

EXAMPLE 7.8

If plane $z = 0$ carries uniform current $\mathbf{K} = K_y\mathbf{a}_y$,

$$\mathbf{H} = \begin{cases} 1/2\, K_y\mathbf{a}_x, & z > 0 \\ -1/2\, K_y\mathbf{a}_x, & z < 0 \end{cases}$$

This was obtained in Section 7.4 using Ampere's law. Obtain this by using the concept of vector magnetic potential.

Solution:

Consider the current sheet as in Figure 7.21. From eq. (7.42),

$$d\mathbf{A} = \frac{\mu_o \mathbf{K}\, dS}{4\pi R}$$

In this problem, $\mathbf{K} = K_y \mathbf{a}_y$, $dS = dx'\, dy'$, and for $z > 0$,

$$\begin{aligned} R = |\mathbf{R}| &= |(0, 0, z) - (x', y', 0)| \\ &= [(x')^2 + (y')^2 + z^2]^{1/2} \end{aligned} \tag{7.8.1}$$

where the primed coordinates are for the source point while the unprimed coordinates are for the field point. It is necessary (and customary) to distinguish between the two points to avoid confusion (see Figure 7.19). Hence

$$d\mathbf{A} = \frac{\mu_o K_y\, dx'\, dy'\, \mathbf{a}_y}{4\pi[(x')^2 + (y')^2 + z^2]^{1/2}}$$

$$\begin{aligned} d\mathbf{B} = \nabla \times d\mathbf{A} &= -\frac{\partial}{\partial z} dA_y\, \mathbf{a}_x \\ &= \frac{\mu_o K_y z\, dx'\, dy'\, \mathbf{a}_x}{4\pi[(x')^2 + (y')^2 + z^2]^{3/2}} \end{aligned}$$

$$\mathbf{B} = \frac{\mu_o K_y z \mathbf{a}_x}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx'\, dy'}{[(x')^2 + (y')^2 + z^2]^{3/2}} \tag{7.8.2}$$

In the integrand, we may change coordinates from Cartesian to cylindrical for convenience so that

$$\begin{aligned} \mathbf{B} &= \frac{\mu_o K_y z \mathbf{a}_x}{4\pi} \int_{\rho'=0}^{\infty} \int_{\phi'=0}^{2\pi} \frac{\rho'\, d\phi'\, d\rho'}{[(\rho')^2 + z^2]^{3/2}} \\ &= \frac{\mu_o K_y z \mathbf{a}_x}{4\pi} 2\pi \int_0^{\infty} [(\rho')^2 + z^2]^{-3/2}\, 1/2\, d[(\rho')^2] \\ &= \frac{\mu_o K_y z \mathbf{a}_x}{2} \left. \frac{-1}{[(\rho')^2 + z^2)^{1/2}} \right|_{\rho'=0}^{\infty} \\ &= \frac{\mu_o K_y \mathbf{a}_x}{2} \end{aligned}$$

Hence

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_o} = \frac{K_y}{2} \mathbf{a}_x, \qquad \text{for } z > 0$$

By simply replacing z by $-z$ in eq. (7.8.2) and following the same procedure, we obtain

$$\mathbf{H} = -\frac{K_y}{2} \mathbf{a}_x, \qquad \text{for } z < 0$$

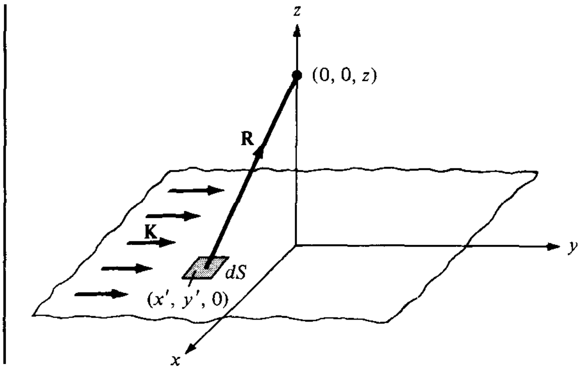


Figure 7.21 For Example 7.8; infinite current sheet.

PRACTICE EXERCISE 7.8

Repeat Example 7.8 by using Biot–Savart's law to determine **H** at points $(0, 0, h)$ and $(0, 0, -h)$.

†7.8 DERIVATION OF BIOT–SAVART'S LAW AND AMPERE'S LAW

Both Biot–Savart's law and Ampere's law may be derived using the concept of magnetic vector potential. The derivation will involve the use of the vector identities in eq. (7.48) and

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{7.52}$$

Since Biot–Savart's law as given in eq. (7.4) is basically on line current, we begin our derivation with eqs. (7.39) and (7.41); that is,

$$\mathbf{B} = \nabla \times \oint_L \frac{\mu_o I \, d\mathbf{l}'}{4\pi R} = \frac{\mu_o I}{4\pi} \oint_L \nabla \times \frac{1}{R} \, d\mathbf{l}', \tag{7.53}$$

where R is as defined in eq. (7.45). If the vector identity in eq. (7.48) is applied by letting $\mathbf{F} = d\mathbf{l}$ and $f = 1/R$, eq. (7.53) becomes

$$\mathbf{B} = \frac{\mu_o I}{4\pi} \oint_L \left[\frac{1}{R} \nabla \times d\mathbf{l}' + \left(\nabla \frac{1}{R} \right) \times d\mathbf{l}' \right] \tag{7.54}$$

Since ∇ operates with respect to (x, y, z) and $d\mathbf{l}'$ is a function of (x', y', z'), $\nabla \times d\mathbf{l}' = 0$. Also

$$\frac{1}{R} = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{-1/2} \tag{7.55}$$

$$\nabla\left[\frac{1}{R}\right] = -\frac{(x - x')\mathbf{a}_x + (y - y')\mathbf{a}_y + (z - z')\mathbf{a}_z}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} = -\frac{\mathbf{a}_R}{R^2} \tag{7.56}$$

where $\mathbf{a}_R$ is a unit vector from the source point to the field point. Thus eq. (7.54) (upon dropping the prime in $d\mathbf{l}'$) becomes

$$\mathbf{B} = \frac{\mu_o I}{4\pi} \oint_L \frac{d\mathbf{l} \times \mathbf{a}_R}{R^2} \tag{7.57}$$

which is Biot–Savart's law.

Using the identity in eq. (7.52) with eq. (7.39), we obtain

$$\nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{7.58}$$

It can be shown that for a static magnetic field

$$\boxed{\nabla \cdot \mathbf{A} = 0} \tag{7.59}$$

so that upon replacing $\mathbf{B}$ with $\mu_o\mathbf{H}$ and using eq. (7.19), eq. (7.58) becomes

$$\nabla^2\mathbf{A} = -\mu_o \nabla \times \mathbf{H}$$

or

$$\boxed{\nabla^2\mathbf{A} = -\mu_o\mathbf{J}} \tag{7.60}$$

which is called the *vector Poisson's equation.* It is similar to Poisson's equation ($\nabla^2 V = -\rho_v/\varepsilon$) in electrostatics. In Cartesian coordinates, eq. (7.60) may be decomposed into three scalar equations:

$$\begin{aligned} \nabla^2 A_x &= -\mu_o J_x \\ \nabla^2 A_y &= -\mu_o J_y \\ \nabla^2 A_z &= -\mu_o J_z \end{aligned} \tag{7.61}$$

which may be regarded as the *scalar Poisson's equations.*

It can also be shown that Ampere's circuit law is consistent with our definition of the magnetic vector potential. From Stokes's theorem and eq. (7.39),

$$\begin{aligned} \oint_L \mathbf{H} \cdot d\mathbf{l} &= \int_S \nabla \times \mathbf{H} \cdot d\mathbf{S} \\ &= \frac{1}{\mu_o} \int_S \nabla \times (\nabla \times \mathbf{A}) \cdot d\mathbf{S} \end{aligned} \tag{7.62}$$

From eqs. (7.52), (7.59), and (7.60),

$$\nabla \times \nabla \times \mathbf{A} = -\nabla^2 \mathbf{A} = \mu_o\mathbf{J}$$

Substituting this into eq. (7.62) yields

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} = I$$

which is Ampere's circuit law.

SUMMARY

1. The basic laws (Biot–Savart's and Ampere's) that govern magnetostatic fields are discussed. Biot–Savart's law, which is similar to Coulomb's law, states that the magnetic field intensity $d\mathbf{H}$ at $\mathbf{r}$ due to current element $I\, d\mathbf{l}$ at $\mathbf{r}'$ is

$$d\mathbf{H} = \frac{I\, d\mathbf{l} \times \mathbf{R}}{4\pi R^3} \qquad \text{(in A/m)}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $R = |\mathbf{R}|$. For surface or volume current distribution, we replace $I\, d\mathbf{l}$ with $\mathbf{K}\, dS$ or $\mathbf{J}\, dv$ respectively; that is,

$$I\, d\mathbf{l} \equiv \mathbf{K}\, dS \equiv \mathbf{J}\, dv$$

2. Ampere's circuit law, which is similar to Gauss's law, states that the circulation of $\mathbf{H}$ around a closed path is equal to the current enclosed by the path; that is,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc} = \int \mathbf{J} \cdot d\mathbf{S}$$

or

$$\nabla \times \mathbf{H} = \mathbf{J} \qquad \text{(third Maxwell's equation to be derived).}$$

When current distribution is symmetric so that an Amperian path (on which $\mathbf{H} = H_\phi \mathbf{a}_\phi$ is constant) can be found, Ampere's law is useful in determining $\mathbf{H}$; that is,

$$H_\phi \oint dl = I_{enc} \qquad \text{or} \qquad H_\phi = \frac{I_{enc}}{\ell}$$

3. The magnetic flux through a surface S is given by

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} \qquad \text{(in Wb)}$$

where $\mathbf{B}$ is the magnetic flux density in Wb/m^2. In free space,

$$\mathbf{B} = \mu_o \mathbf{H}$$

where $\mu_o = 4\pi \times 10^{-7}$ H/m = permeability of free space.

4. Since an isolated or free magnetic monopole does not exist, the net magnetic flux through a closed surface is zero;

$$\Psi = \oint \mathbf{B} \cdot d\mathbf{S} = 0$$

or

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(fourth Maxwell's equation to be derived).}$$

5. At this point, all four Maxwell's equations for static EM fields have been derived, namely:

$$\nabla \cdot \mathbf{D} = \rho_v$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = 0$$
$$\nabla \times \mathbf{H} = \mathbf{J}$$

6. The magnetic scalar potential V_m is defined as

$$\mathbf{H} = -\nabla V_m \qquad \text{if } \mathbf{J} = 0$$

and the magnetic vector potential $\mathbf{A}$ as

$$\mathbf{B} = \nabla \times \mathbf{A}$$

where $\nabla \cdot \mathbf{A} = 0$. With the definition of $\mathbf{A}$, the magnetic flux through a surface S can be found from

$$\Psi = \oint_L \mathbf{A} \cdot d\mathbf{l}$$

where L is the closed path defining surface S (see Figure 3.20). Rather than using Biot–Savart's law, the magnetic field due to a current distribution may be found using $\mathbf{A}$, a powerful approach that is particularly useful in antenna theory. For a current element $I\,d\mathbf{l}$ at $\mathbf{r}'$, the magnetic vector potential at $\mathbf{r}$ is

$$\mathbf{A} = \int \frac{\mu_o I\,d\mathbf{l}}{4\pi R}, \qquad R = |\mathbf{r} - \mathbf{r}'|$$

7. Elements of similarity between electric and magnetic fields exist. Some of these are listed in Table 7.1. Corresponding to Poisson's equation $\nabla^2 V = -\rho_v/\varepsilon$, for example, is

$$\nabla^2 \mathbf{A} = -\mu_o \mathbf{J}$$

EVIEW QUESTIONS

7.1 One of the following is not a source of magnetostatic fields:

(a) A dc current in a wire
(b) A permanent magnet
(c) An accelerated charge
(d) An electric field linearly changing with time
(e) A charged disk rotating at uniform speed

7.2 Identify the configuration in Figure 7.22 that is not a correct representation of I and $\mathbf{H}$.

7.3 Consider points A, B, C, D, and E on a circle of radius 2 as shown in Figure 7.23. The items in the right list are the values of $\mathbf{a}_\phi$ at different points on the circle. Match these items with the points in the list on the left.

(a) A	(i) $\mathbf{a}_x$
(b) B	(ii) $-\mathbf{a}_x$
(c) C	(iii) $\mathbf{a}_y$
(d) D	(iv) $-\mathbf{a}_y$
(e) E	(v) $\dfrac{\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}}$
	(vi) $\dfrac{-\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}}$
	(vii) $\dfrac{-\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}}$
	(viii) $\dfrac{\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}}$

7.4 The z-axis carries filamentary current of 10π A along $\mathbf{a}_z$. Which of these is incorrect?

(a) $\mathbf{H} = -\mathbf{a}_x$ A/m at (0, 5, 0)

(b) $\mathbf{H} = \mathbf{a}_\phi$ A/m at $(5, \pi/4, 0)$

(c) $\mathbf{H} = -0.8\mathbf{a}_x - 0.6\mathbf{a}_y$ at $(-3, 4, 0)$

(d) $\mathbf{H} = -\mathbf{a}_\phi$ at $(5, 3\pi/2, 0)$

7.5 Plane $y = 0$ carries a uniform current of $30\mathbf{a}_z$ mA/m. At $(1, 10, -2)$, the magnetic field intensity is

(a) $-15\mathbf{a}_x$ mA/m

(b) $15\mathbf{a}_x$ mA/m

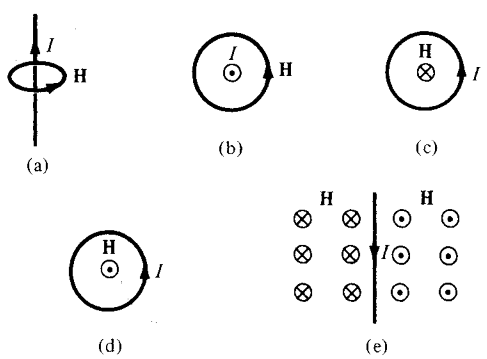


Figure 7.22 For Review Question 7.2.

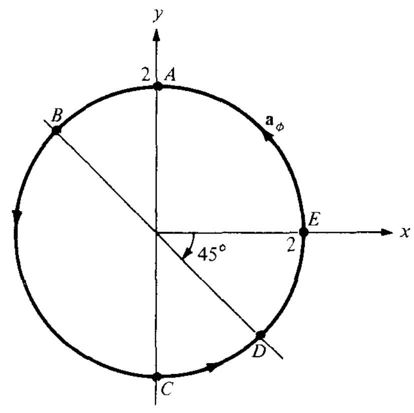


Figure 7.23 For Review Question 7.3.

(c) $477.5\mathbf{a}_y$ μA/m

(d) $18.85\mathbf{a}_y$ nA/m

(e) None of the above

7.6 For the currents and closed paths of Figure 7.24, calculate the value of $\oint_L \mathbf{H} \cdot d\mathbf{l}$.

7.7 Which of these statements is not characteristic of a static magnetic field?

(a) It is solenoidal.

(b) It is conservative.

(c) It has no sinks or sources.

(d) Magnetic flux lines are always closed.

(e) The total number of flux lines entering a given region is equal to the total number of flux lines leaving the region.

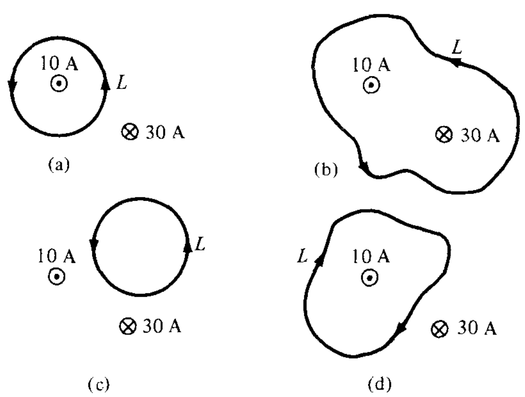


Figure 7.24 For Review Question 7.6.

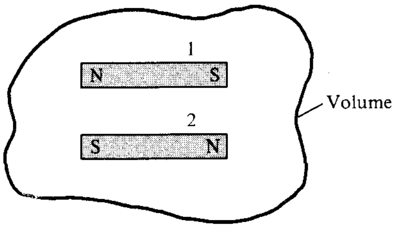


Figure 7.25 For Review Question 7.10.

7.8 Two identical coaxial circular coils carry the same current I but in opposite directions. The magnitude of the magnetic field **B** at a point on the axis midway between the coils is

(a) Zero
(b) The same as that produced by one coil
(c) Twice that produced by one coil
(d) Half that produced by one coil.

7.9 One of these equations is not Maxwell's equation for a static electromagnetic field in a linear homogeneous medium.

(a) $\nabla \cdot \mathbf{B} = 0$
(b) $\nabla \times \mathbf{D} = 0$
(c) $\oint \mathbf{B} \cdot d\mathbf{l} = \mu_o I$
(d) $\oint \mathbf{D} \cdot d\mathbf{S} = Q$
(e) $\nabla^2 \mathbf{A} = \mu_o \mathbf{J}$

7.10 Two bar magnets with their north poles have strength $Q_{m1} = 20$ A · m and $Q_{m2} = 10$ A · m (magnetic charges) are placed inside a volume as shown in Figure 7.25. The magnetic flux leaving the volume is

(a) 200 Wb
(b) 30 Wb
(c) 10 Wb
(d) 0 Wb
(e) -10 Wb

Answers: 7.1c, 7.2c, 7.3 (a)-(ii), (b)-(vi), (c)-(i), (d)-(v), (e)-(iii), 7.4d, 7.5a, 7.6 (a) 10 A, (b) -20 A, (c) 0, (d) -10 A, 7.7b, 7.8a, 7.9e, 7.10d.

PROBLEMS

7.1 (a) State Biot–Savart's law

(b) The y- and z-axes, respectively, carry filamentary currents 10 A along $\mathbf{a}_y$ and 20 A along $-\mathbf{a}_z$. Find **H** at $(-3, 4, 5)$.

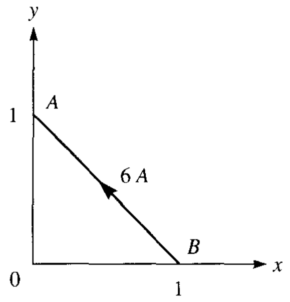


Figure 7.26 For Problem 7.3.

7.2 A conducting filament carries current I from point $A(0, 0, a)$ to point $B(0, 0, b)$. Show that at point $P(x, y, 0)$,

$$\mathbf{H} = \frac{I}{4\pi\sqrt{x^2 + y^2}}\left[\frac{b}{\sqrt{x^2 + y^2 + b^2}} - \frac{a}{\sqrt{x^2 + y^2 + a^2}}\right]\mathbf{a}_y$$

7.3 Consider AB in Figure 7.26 as part of an electric circuit. Find $\mathbf{H}$ at the origin due to AB.

7.4 Repeat Problem 7.3 for the conductor AB in Figure 7.27.

7.5 Line $x = 0, y = 0, 0 \leq z \leq 10$ m carries current 2 A along $\mathbf{a}_z$. Calculate $\mathbf{H}$ at points

(a) $(5, 0, 0)$
(b) $(5, 5, 0)$
(c) $(5, 15, 0)$
(d) $(5, -15, 0)$

***7.6** (a) Find $\mathbf{H}$ at $(0, 0, 5)$ due to side 2 of the triangular loop in Figure 7.6(a).
(b) Find $\mathbf{H}$ at $(0, 0, 5)$ due to the entire loop.

7.7 An infinitely long conductor is bent into an L shape as shown in Figure 7.28. If a direct current of 5 A flows in the current, find the magnetic field intensity at (a) $(2, 2, 0)$, (b) $(0, -2, 0)$, and (c) $(0, 0, 2)$.

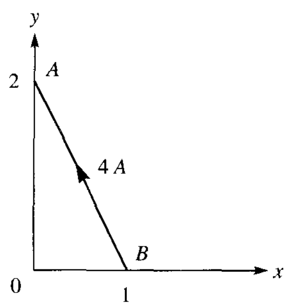


Figure 7.27 For Problem 7.4.

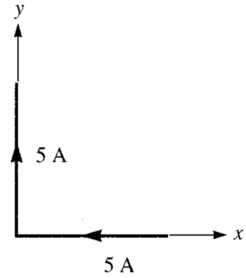


Figure 7.28 Current filament for Problem 7.7.

7.8 Find **H** at the center C of an equilateral triangular loop of side 4 m carrying 5 A of current as in Figure 7.29.

7.9 A rectangular loop carrying 10 A of current is placed on $z = 0$ plane as shown in Figure 7.30. Evaluate **H** at

(a) (2, 2, 0)
(b) (4, 2, 0)
(c) (4, 8, 0)
(d) (0, 0, 2)

7.10 A square conducting loop of side $2a$ lies in the $z = 0$ plane and carries a current I in the counterclockwise direction. Show that at the center of the loop

$$\mathbf{H} = \frac{\sqrt{2}I}{\pi a}\,\mathbf{a}_z$$

***7.11** (a) A filamentary loop carrying current I is bent to assume the shape of a regular polygon of n sides. Show that at the center of the polygon

$$H = \frac{nI}{2\pi r}\sin\frac{\pi}{n}$$

where r is the radius of the circle circumscribed by the polygon.

(b) Apply this to cases when $n = 3$ and $n = 4$ and see if your results agree with those for the triangular loop of Problem 7.8 and the square loop of Problem 7.10, respectively.

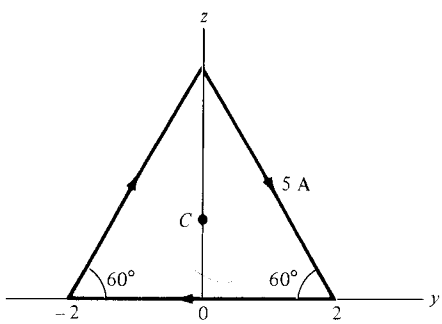


Figure 7.29 Equilateral triangular loop for Problem 7.8.

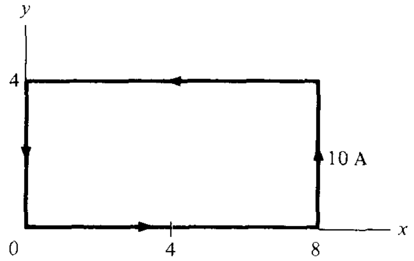


Figure 7.30 Rectangular loop of Problem 7.9.

(c) As n becomes large, show that the result of part (a) becomes that of the circular loop of Example 7.3.

7.12 For the filamentary loop shown in Figure 7.31, find the magnetic field strength at O.

7.13 Two identical current loops have their centers at (0, 0, 0) and (0, 0, 4) and their axes the same as the z-axis (so that the "Helmholtz coil" is formed). If each loop has radius 2 m and carries current 5 A in $\mathbf{a}_\phi$, calculate $\mathbf{H}$ at

(a) (0, 0, 0)
(b) (0, 0, 2)

7.14 A 3-cm-long solenoid carries a current of 400 mA. If the solenoid is to produce a magnetic flux density of 5 mWb/m^2, how many turns of wire are needed?

7.15 A solenoid of radius 4 mm and length 2 cm has 150 turns/m and carries current 500 mA. Find: (a) $|\mathbf{H}|$ at the center, (b) $|\mathbf{H}|$ at the ends of the solenoid.

7.16 Plane $x = 10$ carries current 100 mA/m along $\mathbf{a}_z$ while line $x = 1$, $y = -2$ carries filamentary current 20π mA along $\mathbf{a}_z$. Determine $\mathbf{H}$ at (4, 3, 2).

7.17 (a) State Ampere's circuit law.

(b) A hollow conducting cylinder has inner radius a and outer radius b and carries current I along the positive z-direction. Find $\mathbf{H}$ everywhere.

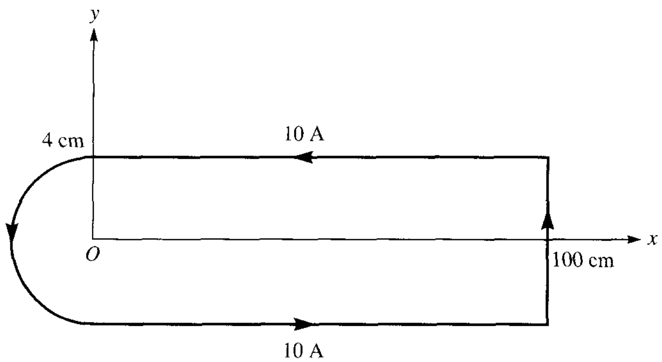


Figure 7.31 Filamentary loop of Problem 7.12; not drawn to scale.

7.18 (a) An infinitely long solid conductor of radius a is placed along the z-axis. If the conductor carries current I in the $+z$ direction, show that

$$\mathbf{H} = \frac{I\rho}{2\pi a^2}\,\mathbf{a}_\phi$$

within the conductor. Find the corresponding current density.

(b) If $I = 3$ A and $a = 2$ cm in part (a), find $\mathbf{H}$ at (0, 1 cm, 0) and (0, 4 cm, 0).

7.19 If $\mathbf{H} = y\mathbf{a}_x - x\mathbf{a}_y$ A/m on plane $z = 0$, (a) determine the current density and (b) verify Ampere's law by taking the circulation of $\mathbf{H}$ around the edge of the rectangle $z = 0, 0 < x < 3, -1 < y < 4$.

7.20 In a certain conducting region,

$$\mathbf{H} = yz(x^2 + y^2)\mathbf{a}_x - y^2xz\mathbf{a}_y + 4x^2y^2\mathbf{a}_z \text{ A/m}$$

(a) Determine $\mathbf{J}$ at $(5, 2, -3)$

(b) Find the current passing through $x = -1, 0 < y, z < 2$

(c) Show that $\nabla \cdot \mathbf{B} = 0$

7.21 An infinitely long filamentary wire carries a current of 2 A in the $+z$-direction. Calculate

(a) $\mathbf{B}$ at $(-3, 4, 7)$

(b) The flux through the square loop described by $2 \leq \rho \leq 6, 0 \leq z \leq 4, \phi = 90°$

7.22 The electric motor shown in Figure 7.32 has field

$$\mathbf{H} = \frac{10^6}{\rho}\sin 2\phi\,\mathbf{a}_\rho \text{ A/m}$$

Calculate the flux per pole passing through the air gap if the axial length of the pole is 20 cm.

7.23 Consider the two-wire transmission line whose cross section is illustrated in Figure 7.33. Each wire is of radius 2 cm and the wires are separated 10 cm. The wire centered at (0, 0)

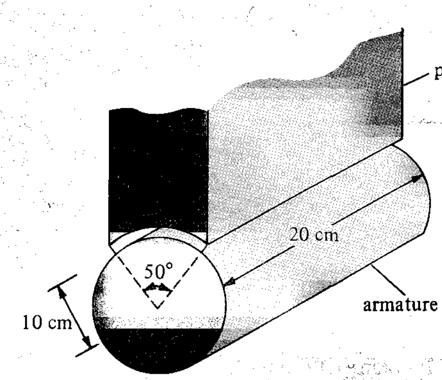


Figure 7.32 Electric motor pole of Problem 7.22.

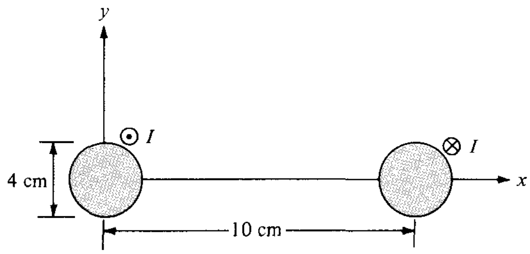


Figure 7.33 Two-wire line of Problem 7.23.

carries current 5 A while the other centered at (10 cm, 0) carries the return current. Find **H** at

(a) (5 cm, 0)

(b) (10 cm, 5 cm)

7.24 Determine the magnetic flux through a rectangular loop ($a \times b$) due to an infinitely long conductor carrying current I as shown in Figure 7.34. The loop and the straight conductors are separated by distance d.

***7.25** A brass ring with triangular cross section encircles a very long straight wire concentrically as in Figure 7.35. If the wire carries a current I, show that the total number of magnetic flux lines in the ring is

$$\Psi = \frac{\mu_o I h}{2\pi b}\left[b - a \ln \frac{a+b}{b}\right]$$

Calculate Ψ if $a = 30$ cm, $b = 10$ cm, $h = 5$ cm, and $I = 10$ A.

7.26 Consider the following arbitrary fields. Find out which of them can possibly represent electrostatic or magnetostatic field in free space.

(a) $\mathbf{A} = y \cos ax\mathbf{a}_x + (y + e^{-x})\mathbf{a}_z$

(b) $\mathbf{B} = \dfrac{20}{\rho}\mathbf{a}_\rho$

(c) $\mathbf{C} = r^2 \sin\theta\, \mathbf{a}_\phi$

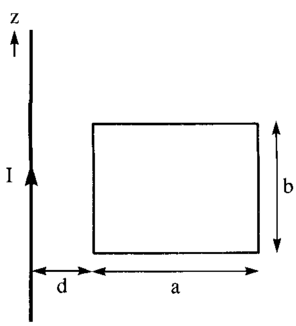


Figure 7.34 For Problem 7.24

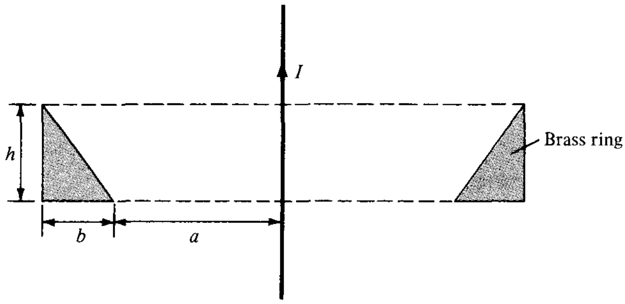


Figure 7.35 Cross section of a brass ring enclosing a long straight wire; for Problem 7.25.

7.27 Reconsider the previous problem for the following fields.

(a) $\mathbf{D} = y^2 z \mathbf{a}_x + 2(x + 1)yz \mathbf{a}_y - (x + 1)z^2 \mathbf{a}_z$

(b) $\mathbf{E} = \dfrac{(z + 1)}{\rho} \cos \phi \, \mathbf{a}_\rho + \dfrac{\sin \phi}{\rho}$

(c) $\mathbf{F} = \dfrac{1}{r^2} (2 \cos \theta \, \mathbf{a}_r + \sin \theta \, \mathbf{a}_\theta)$

7.28 For a current distribution in free space,

$$\mathbf{A} = (2x^2 y + yz)\mathbf{a}_x + (xy^2 - xz^3)\mathbf{a}_y - (6xyz - 2x^2y^2)\mathbf{a}_z \text{ Wb/m}$$

(a) Calculate **B**.

(b) Find the magnetic flux through a loop described by $x = 1, 0 < y, z < 2$.

(c) Show that $\nabla \cdot \mathbf{A} = 0$ and $\nabla \cdot \mathbf{B} = 0$.

7.29 The magnetic vector potential of a current distribution in free space is given by

$$\mathbf{A} = 15e^{-\rho} \sin \phi \, \mathbf{a}_z \text{ Wb/m}$$

Find **H** at $(3, \pi/4, -10)$. Calculate the flux through $\rho = 5, 0 \leq \phi \leq \pi/2, 0 \leq z \leq 10$.

7.30 A conductor of radius a carries a uniform current with $\mathbf{J} = J_o \mathbf{a}_z$. Show that the magnetic vector potential for $\rho > a$ is

$$\mathbf{A} = -\frac{1}{4} \mu_o J_o \rho^2 \mathbf{a}_z$$

7.31 An infinitely long conductor of radius a is placed such that its axis is along the z-axis. The vector magnetic potential, due to a direct current I_o flowing along $\mathbf{a}_z$ in the conductor, is given by

$$\mathbf{A} = \frac{-I_o}{4\pi a^2} \mu_o (x^2 + y^2) \, \mathbf{a}_z \text{ Wb/m}$$

Find the corresponding **H**. Confirm your result using Ampere's law.

7.32 The magnetic vector potential of two parallel infinite straight current filaments in free space carrying equal current I in opposite direction is

$$\mathbf{A} = \frac{\mu I}{2\pi} \ln \frac{d - \rho}{\rho} \mathbf{a}_z$$

where d is the separation distance between the filaments (with one filament placed along the z-axis). Find the corresponding magnetic flux density **B**.

7.33 Find the current density **J** to

$$\mathbf{A} = \frac{10}{\rho^2} \mathbf{a}_z \text{ Wb/m}$$

in free space.

7.34 Prove that the magnetic scalar potential at $(0, 0, z)$ due to a circular loop of radius a shown in Figure 7.8(a) is

$$V_m = \frac{I}{2}\left[1 - \frac{z}{[z^2 + a^2]^{1/2}}\right]$$

***7.35** A coaxial transmission line is constructed such that the radius of the inner conductor is a and the outer conductor has radii $3a$ and $4a$. Find the vector magnetic potential within the outer conductor. Assume $A_z = 0$ for $\rho = 3a$.

7.36 The z-axis carries a filamentary current 12 A along $\mathbf{a}_z$. Calculate V_m at $(4, 30°, -2)$ if $V_m = 0$ at $(10, 60°, 7)$.

7.37 Plane $z = -2$ carries a current of $50\mathbf{a}_y$ A/m. If $V_m = 0$ at the origin, find V_m at

(a) $(-2, 0, 5)$

(b) $(10, 3, 1)$

7.38 Prove in cylindrical coordinates that

(a) $\nabla \times (\nabla V) = 0$

(b) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

7.39 If $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $R = |\mathbf{R}|$, show that

$$\nabla \frac{1}{\mathbf{R}} = -\nabla' \frac{1}{\mathbf{R}} = -\frac{\mathbf{R}}{R^3}$$

where ∇ and ∇' are del operators with respect to (x, y, z) and (x', y', z), respectively.

Chapter 8

MAGNETIC FORCES, MATERIALS, AND DEVICES

Do all the good you can,
By all the means you can,
In all the ways you can,
In all the places you can,
At all the times you can,
To all the people you can,
As long as ever you can.

—JOHN WESLEY

8.1 INTRODUCTION

Having considered the basic laws and techniques commonly used in calculating magnetic field **B** due to current-carrying elements, we are prepared to study the force a magnetic field exerts on charged particles, current elements, and loops. Such a study is important to problems on electrical devices such as ammeters, voltmeters, galvanometers, cyclotrons, plasmas, motors, and magnetohydrodynamic generators. The precise definition of the magnetic field, deliberately sidestepped in the previous chapter, will be given here. The concepts of magnetic moments and dipole will also be considered.

Furthermore, we will consider magnetic fields in material media, as opposed to the magnetic fields in vacuum or free space examined in the previous chapter. The results of the preceding chapter need only some modification to account for the presence of materials in a magnetic field. Further discussions will cover inductors, inductances, magnetic energy, and magnetic circuits.

8.2 FORCES DUE TO MAGNETIC FIELDS

There are at least three ways in which force due to magnetic fields can be experienced. The force can be (a) due to a moving charged particle in a **B** field, (b) on a current element in an external **B** field, or (c) between two current elements.

A. Force on a Charged Particle

According to our discussion in Chapter 4, the electric force $\mathbf{F}_e$ on a stationary or moving electric charge Q in an electric field is given by Coulomb's experimental law and is related to the electric field intensity $\mathbf{E}$ as

$$\mathbf{F}_e = Q\mathbf{E} \tag{8.1}$$

This shows that if Q is positive, $\mathbf{F}_e$ and $\mathbf{E}$ have the same direction.

A magnetic field can exert force only on a moving charge. From experiments, it is found that the magnetic force $\mathbf{F}_m$ experienced by a charge Q moving with a velocity $\mathbf{u}$ in a magnetic field $\mathbf{B}$ is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B} \tag{8.2}$$

This clearly shows that $\mathbf{F}_m$ is perpendicular to both $\mathbf{u}$ and $\mathbf{B}$.

From eqs. (8.1) and (8.2), a comparison between the electric force $\mathbf{F}_e$ and the magnetic force $\mathbf{F}_m$ can be made. $\mathbf{F}_e$ is independent of the velocity of the charge and can perform work on the charge and change its kinetic energy. Unlike $\mathbf{F}_e$, $\mathbf{F}_m$ depends on the charge velocity and is normal to it. $\mathbf{F}_m$ cannot perform work because it is at right angles to the direction of motion of the charge ($\mathbf{F}_m \cdot d\mathbf{l} = 0$); it does not cause an increase in kinetic energy of the charge. The magnitude of $\mathbf{F}_m$ is generally small compared to $\mathbf{F}_e$ except at high velocities.

For a moving charge Q in the presence of both electric and magnetic fields, the total force on the charge is given by

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m$$

or

$$\boxed{\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})} \tag{8.3}$$

This is known as the *Lorentz force equation.*[1] It relates mechanical force to electrical force. If the mass of the charged particle moving in $\mathbf{E}$ and $\mathbf{B}$ fields is m, by Newton's second law of motion.

$$\mathbf{F} = m\frac{d\mathbf{u}}{dt} = Q\,(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{8.4}$$

The solution to this equation is important in determining the motion of charged particles in $\mathbf{E}$ and $\mathbf{B}$ fields. We should bear in mind that in such fields, energy transfer can be only by means of the electric field. A summary on the force exerted on a charged particle is given in Table 8.1.

Since eq. (8.2) is closely parallel to eq. (8.1), which defines the electric field, some authors and instructors prefer to begin their discussions on magnetostatics from eq. (8.2) just as discussions on electrostatics usually begin with Coulomb's force law.

[1]After Hendrik Lorentz (1853–1928), who first applied the equation to electric field motion.