is associated with Maxwell's equations. Also the equation of continuity

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \tag{9.29}$$

is implicit in Maxwell's equations. The concepts of linearity, isotropy, and homogeneity of a material medium still apply for time-varying fields; in a linear, homogeneous, and isotropic medium characterized by σ, ε, and μ, the constitutive relations

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_o\mathbf{E} + \mathbf{P} \tag{9.30a}$$

$$\mathbf{B} = \mu\mathbf{H} = \mu_o(\mathbf{H} + \mathbf{M}) \tag{9.30b}$$

$$\mathbf{J} = \sigma\mathbf{E} + \rho_v\mathbf{u} \tag{9.30c}$$

hold for time-varying fields. Consequently, the boundary conditions

$$E_{1t} = E_{2t} \quad \text{or} \quad (\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{a}_{n12} = 0 \tag{9.31a}$$

$$H_{1t} - H_{2t} = K \quad \text{or} \quad (\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K} \tag{9.31b}$$

$$D_{1n} - D_{2n} = \rho_s \quad \text{or} \quad (\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{a}_{n12} = \rho_s \tag{9.31c}$$

$$B_{1n} - B_{2n} = 0 \quad \text{or} \quad (\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{a}_{n12} = 0 \tag{9.31d}$$

remain valid for time-varying fields. However, for a perfect conductor ($\sigma \simeq \infty$) in a time-varying field,

$$\mathbf{E} = 0, \qquad \mathbf{H} = 0, \qquad \mathbf{J} = 0 \tag{9.32}$$

and hence,

$$\mathbf{B}_n = 0, \qquad \mathbf{E}_t = 0 \tag{9.33}$$

For a perfect dielectric ($\sigma \simeq 0$), eq. (9.31) holds except that $\mathbf{K} = 0$. Though eqs. (9.28) to (9.33) are not Maxwell's equations, they are associated with them.

To complete this summary section, we present a structure linking the various potentials and vector fields of the electric and magnetic fields in Figure 9.11. This electromagnetic flow diagram helps with the visualization of the basic relationships between field quantities. It also shows that it is usually possible to find alternative formulations, for a given problem, in a relatively simple manner. It should be noted that in Figures 9.10(b) and (c), we introduce ρ^m as the free magnetic density (similar to ρ_v), which is, of course, zero, $\mathbf{A}_e$ as the magnetic current density (analogous to $\mathbf{J}$). Using terms from stress analysis, the principal relationships are typified as:

(a) compatibility equations

$$\nabla \cdot \mathbf{B} = \rho^m = 0 \tag{9.34}$$

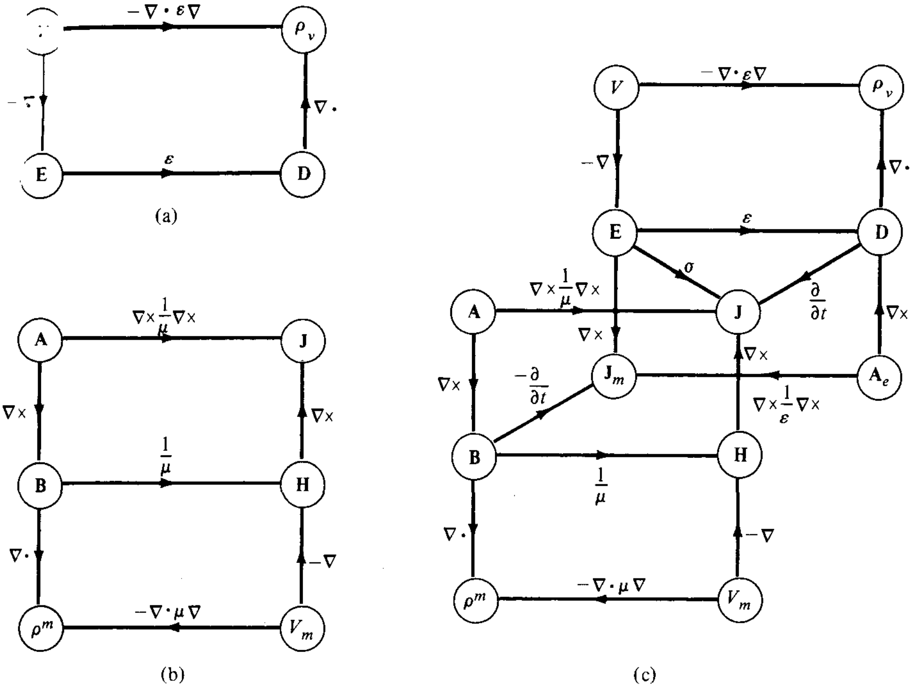

Figure 9.11 Electromagnetic flow diagram showing the relationship between the potentials and vector fields: **(a)** electrostatic system, **(b)** magnetostatic system, **(c)** electromagnetic system. [Adapted with permission from IEE Publishing Dept.]

and

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = \mathbf{J}_m \tag{9.35}$$

(b) constitutive equations

$$\mathbf{B} = \mu \mathbf{H} \tag{9.36}$$

and

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{9.37}$$

(c) equilibrium equations

$$\nabla \cdot \mathbf{D} = \rho_v \tag{9.38}$$

and

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{9.39}$$

9.6 TIME-VARYING POTENTIALS

For static EM fields, we obtained the electric scalar potential as

$$V = \int_v \frac{\rho_v \, dv}{4\pi\varepsilon R} \tag{9.40}$$

and the magnetic vector potential as

$$\mathbf{A} = \int_v \frac{\mu J \, dv}{4\pi R} \tag{9.41}$$

We would like to examine what happens to these potentials when the fields are time varying. Recall that $\mathbf{A}$ was defined from the fact that $\nabla \cdot \mathbf{B} = 0$, which still holds for time-varying fields. Hence the relation

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \tag{9.42}$$

holds for time-varying situations. Combining Faraday's law in eq. (9.8) with eq. (9.42) gives

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) \tag{9.43a}$$

or

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \tag{9.43b}$$

Since the curl of the gradient of a scalar field is identically zero (see Practice Exercise 3.10), the solution to eq. (9.43b) is

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V \tag{9.44}$$

or

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}} \tag{9.45}$$

From eqs. (9.42) and (9.45), we can determine the vector fields $\mathbf{B}$ and $\mathbf{E}$ provided that the potentials $\mathbf{A}$ and V are known. However, we still need to find some expressions for $\mathbf{A}$ and V similar to those in eqs. (9.40) and (9.41) that are suitable for time-varying fields.

From Table 9.1 or eq. (9.38) we know that $\nabla \cdot \mathbf{D} = \rho_v$ is valid for time-varying conditions. By taking the divergence of eq. (9.45) and making use of eqs. (9.37) and (9.38), we obtain

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon} = -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A})$$

or

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho_v}{\varepsilon} \tag{9.46}$$

Taking the curl of eq. (9.42) and incorporating eqs. (9.23) and (9.45) results in

$$\begin{aligned}\nabla \times \nabla \times \mathbf{A} &= \mu \mathbf{J} + \varepsilon\mu \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right) \\ &= \mu \mathbf{J} - \mu\varepsilon\, \nabla\left(\frac{\partial V}{\partial t}\right) - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}\end{aligned} \tag{9.47}$$

where $\mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$ have been assumed. By applying the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{9.48}$$

to eq. (9.47),

$$\nabla^2 \mathbf{A} - \nabla(\nabla \cdot \mathbf{A}) = -\mu \mathbf{J} + \mu\varepsilon\, \nabla\left(\frac{\partial V}{\partial t}\right) + \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{9.49}$$

A vector field is uniquely defined when its curl and divergence are specified. The curl of **A** has been specified by eq. (9.42); for reasons that will be obvious shortly, we may choose the divergence of **A** as

$$\boxed{\nabla \cdot \mathbf{A} = -\mu\varepsilon \frac{\partial V}{\partial t}} \tag{9.50}$$

This choice relates **A** and V and it is called the *Lorentz condition for potentials.* We had this in mind when we chose $\nabla \cdot \mathbf{A} = 0$ for magnetostatic fields in eq. (7.59). By imposing the Lorentz condition of eq. (9.50), eqs. (9.46) and (9.49), respectively, become

$$\boxed{\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon}} \tag{9.51}$$

and

$$\boxed{\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}} \tag{9.52}$$

which are *wave equations* to be discussed in the next chapter. The reason for choosing the Lorentz condition becomes obvious as we examine eqs. (9.51) and (9.52). It uncouples eqs. (9.46) and (9.49) and also produces a symmetry between eqs. (9.51) and (9.52). It can be shown that the Lorentz condition can be obtained from the continuity equation; therefore, our choice of eq. (9.50) is not arbitrary. Notice that eqs. (6.4) and (7.60) are special static cases of eqs. (9.51) and (9.52), respectively. In other words, potentials V and **A** satisfy Poisson's equations for time-varying conditions. Just as eqs. (9.40) and (9.41) are the solutions, or the integral forms of eqs. (6.4) and (7.60), it can be shown that the solutions[5] to eqs. (9.51) and (9.52) are

$$V = \int_v \frac{[\rho_v]\, dv}{4\pi\varepsilon R} \tag{9.53}$$

and

$$\mathbf{A} = \int_v \frac{\mu[\mathbf{J}]\, dv}{4\pi R} \tag{9.54}$$

The term $[\rho_v]$ (or $[\mathbf{J}]$) means that the time t in $\rho_v(x, y, z, t)$ [or $\mathbf{J}(x, y, z, t)$] is replaced by the *retarded time* t' given by

$$t' = t - \frac{R}{\mu} \tag{9.55}$$

where $R = |\mathbf{r} - \mathbf{r}'|$ is the distance between the source point $\mathbf{r}'$ and the observation point $\mathbf{r}$ and

$$u = \frac{1}{\sqrt{\mu\varepsilon}} \tag{9.56}$$

is the velocity of wave propagation. In free space, $u = c \simeq 3 \times 10^8$ m/s is the speed of light in a vacuum. Potentials V and $\mathbf{A}$ in eqs. (9.53) and (9.54) are, respectively, called the *retarded electric scalar potential* and the *retarded magnetic vector potential.* Given ρ_v and $\mathbf{J}$, V and $\mathbf{A}$ can be determined using eqs. (9.53) and (9.54); from V and $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{B}$ can be determined using eqs. (9.45) and (9.42), respectively.

9.7 TIME-HARMONIC FIELDS

So far, our time dependence of EM fields has been arbitrary. To be specific, we shall assume that the fields are *time harmonic.*

A **time-harmonic field** is one that varies periodically or sinusoidally with time.

Not only is sinusoidal analysis of practical value, it can be extended to most waveforms by Fourier transform techniques. Sinusoids are easily expressed in phasors, which are more convenient to work with. Before applying phasors to EM fields, it is worthwhile to have a brief review of the concept of phasor.

A *phasor* z is a complex number that can be written as

$$z = x + jy = r \angle \phi \tag{9.57}$$

[5] For example, see D. K. Cheng, *Field and Wave Electromagnetics.* Reading, MA: Addison-Wesley, 1983, pp. 291–292.

or

$$z = r\,e^{j\phi} = r\,(\cos\phi + j\sin\phi) \tag{9.58}$$

where $j = \sqrt{-1}$, x is the real part of z, y is the imaginary part of z, r is the magnitude of z, given by

$$r = |z| = \sqrt{x^2 + y^2} \tag{9.59}$$

and ϕ is the phase of z, given by

$$\phi = \tan^{-1}\frac{y}{x} \tag{9.60}$$

Here x, y, z, r, and ϕ should not be mistaken as the coordinate variables although they look similar (different letters could have been used but it is hard to find better ones). The phasor z can be represented in *rectangular form* as $z = x + jy$ or in *polar form* as $z = r\,\underline{/\phi} = r\,e^{j\phi}$. The two forms of representing z are related in eqs. (9.57) to (9.60) and illustrated in Figure 9.12. Addition and subtraction of phasors are better performed in rectangular form; multiplication and division are better done in polar form.

Given complex numbers

$$z = x + jy = r\,\underline{/\phi}, \qquad z_1 = x_1 + jy_1 = r_1\,\underline{/\phi_1}, \qquad \text{and} \qquad z_2 = x_2 + jy_2 = r_2\,\underline{/\phi_2}$$

the following basic properties should be noted.

Addition:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \tag{9.61a}$$

Subtraction:

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \tag{9.61b}$$

Multiplication:

$$z_1 z_2 = r_1 r_2\,\underline{/\phi_1 + \phi_2} \tag{9.61c}$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\underline{/\phi_1 - \phi_2} \tag{9.61d}$$

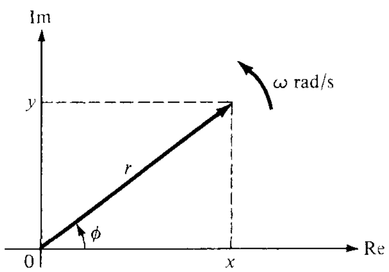

Figure 9.12 Representation of a phasor $z = x + jy = r\,\underline{/\phi}$.

Square Root:

$$\sqrt{z} = \sqrt{r} \,\underline{/\phi/2} \tag{9.61e}$$

Complex Conjugate:

$$z^* = x - jy = r\underline{/-\phi} = re^{-j\phi} \tag{9.61f}$$

Other properties of complex numbers can be found in Appendix A.2.

To introduce the time element, we let

$$\phi = \omega t + \theta \tag{9.62}$$

where θ may be a function of time or space coordinates or a constant. The real (Re) and imaginary (Im) parts of

$$re^{j\phi} = re^{j\theta}e^{j\omega t} \tag{9.63}$$

are, respectively, given by

$$\text{Re}\,(re^{j\phi}) = r\cos\,(\omega t + \theta) \tag{9.64a}$$

and

$$\text{Im}\,(re^{j\phi}) = r\sin\,(\omega t + \theta) \tag{9.64b}$$

Thus a sinusoidal current $I(t) = I_o\cos(\omega t + \theta)$, for example, equals the real part of $I_o e^{j\theta}e^{j\omega t}$. The current $I'(t) = I_o\sin(\omega t + \theta)$, which is the imaginary part of $I_o e^{j\theta}e^{j\omega t}$, can also be represented as the real part of $I_o e^{j\theta}e^{j\omega t}e^{-j90°}$ because $\sin\alpha = \cos(\alpha - 90°)$. However, in performing our mathematical operations, we must be consistent in our use of either the real part or the imaginary part of a quantity but not both at the same time.

The complex term $I_o e^{j\theta}$, which results from dropping the time factor $e^{j\omega t}$ in $I(t)$, is called the *phasor* current, denoted by I_s; that is,

$$I_s = I_o e^{j\theta} = I_0 \,\underline{/\theta} \tag{9.65}$$

where the subscript s denotes the phasor form of $I(t)$. Thus $I(t) = I_o\cos(\omega t + \theta)$, the *instantaneous form*, can be expressed as

$$I(t) = \text{Re}\,(I_s e^{j\omega t}) \tag{9.66}$$

In general, a phasor could be scalar or vector. If a vector $\mathbf{A}(x, y, z, t)$ is a time-harmonic field, the *phasor form* of $\mathbf{A}$ is $\mathbf{A}_s(x, y, z)$; the two quantities are related as

$$\boxed{\mathbf{A} = \text{Re}\,(\mathbf{A}_s e^{j\omega t})} \tag{9.67}$$

For example, if $\mathbf{A} = A_o\cos\,(\omega t - \beta x)\,\mathbf{a}_y$, we can write $\mathbf{A}$ as

$$\mathbf{A} = \text{Re}\,(A_o e^{-j\beta x}\,\mathbf{a}_y e^{j\omega t}) \tag{9.68}$$

Comparing this with eq. (9.67) indicates that the phasor form of $\mathbf{A}$ is

$$\mathbf{A}_s = A_o e^{-j\beta x}\,\mathbf{a}_y \tag{9.69}$$

Notice from eq. (9.67) that

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial t} &= \frac{\partial}{\partial t}\operatorname{Re}(\mathbf{A}_s e^{j\omega t}) \\ &= \operatorname{Re}(j\omega \mathbf{A}_s e^{j\omega t})\end{aligned} \tag{9.70}$$

showing that taking the time derivative of the instantaneous quantity is equivalent to multiplying its phasor form by $j\omega$. That is,

$$\frac{\partial \mathbf{A}}{\partial t} \rightarrow j\omega \mathbf{A}_s \tag{9.71}$$

Similarly,

$$\int \mathbf{A}\, \partial t \rightarrow \frac{\mathbf{A}_s}{j\omega} \tag{9.72}$$

Note that the real part is chosen in eq. (9.67) as in circuit analysis; the imaginary part could equally have been chosen. Also notice the basic difference between the instantaneous form $\mathbf{A}(x, y, z, t)$ and its phasor form $\mathbf{A}_s(x, y, z)$; the former is time dependent and real whereas the latter is time invariant and generally complex. It is easier to work with $\mathbf{A}_s$ and obtain $\mathbf{A}$ from $\mathbf{A}_s$ whenever necessary using eq. (9.67).

We shall now apply the phasor concept to time-varying EM fields. The fields quantities $\mathbf{E}(x, y, z, t)$, $\mathbf{D}(x, y, z, t)$, $\mathbf{H}(x, y, z, t)$, $\mathbf{B}(x, y, z, t)$, $\mathbf{J}(x, y, z, t)$, and $\rho_v(x, y, z, t)$ and their derivatives can be expressed in phasor form using eqs. (9.67) and (9.71). In phasor form, Maxwell's equations for time-harmonic EM fields in a linear, isotropic, and homogeneous medium are presented in Table 9.2. From Table 9.2, note that the time factor $e^{j\omega t}$ disappears because it is associated with every term and therefore factors out, resulting in time-independent equations. Herein lies the justification for using phasors; the time factor can be suppressed in our analysis of time-harmonic fields and inserted when necessary. Also note that in Table 9.2, the time factor $e^{j\omega t}$ has been assumed. It is equally possible to have assumed the time factor $e^{-j\omega t}$, in which case we would need to replace every j in Table 9.2 with $-j$.

TABLE 9.2 Time-Harmonic Maxwell's Equations Assuming Time Factor $e^{j\omega t}$

Point Form	Integral Form
$\nabla \cdot \mathbf{D}_s = \rho_{vs}$	$\oint \mathbf{D}_s \cdot d\mathbf{S} = \int \rho_{vs}\, dv$
$\nabla \cdot \mathbf{B}_s = 0$	$\oint \mathbf{B}_s \cdot d\mathbf{S} = 0$
$\nabla \times \mathbf{E}_s = -j\omega \mathbf{B}_s$	$\oint \mathbf{E}_s \cdot d\mathbf{l} = -j\omega \int \mathbf{B}_s \cdot d\mathbf{S}$
$\nabla \times \mathbf{H}_s = \mathbf{J}_s + j\omega \mathbf{D}_s$	$\oint \mathbf{H}_s \cdot d\mathbf{l} = \int (\mathbf{J}_s + j\omega \mathbf{D}_s) \cdot d\mathbf{S}$

EXAMPLE 9.5 Evaluate the complex numbers

(a) $z_1 = \dfrac{j(3 - j4)^*}{(-1 + j6)(2 + j)^2}$

(b) $z_2 = \left[\dfrac{1 + j}{4 - j8}\right]^{1/2}$

Solution:

(a) This can be solved in two ways: working with z in rectangular form or polar form.

Method 1: (working in rectangular form):
Let

$$z_1 = \frac{z_3 z_4}{z_5 z_6}$$

where

$$z_3 = j$$

$$z_4 = (3 - j4)^* = \text{the complex conjugate of } (3 - j4) = 3 + j4$$

(To find the complex conjugate of a complex number, simply replace every j with $-j$.)

$$z_5 = -1 + j6$$

and

$$z_6 = (2 + j)^2 = 4 - 1 + j4 = 3 + j4$$

Hence,

$$z_3 z_4 = j(3 + j4) = -4 + j3$$

$$z_5 z_6 = (-1 + j6)(3 + j4) = -3 - j4 + j18 - 24 = -27 + j14$$

and

$$z_1 = \frac{-4 + j3}{-27 + j14}$$

Multiplying and dividing z_1 by $-27 - j14$ (rationalization), we have

$$z_1 = \frac{(-4 + j3)(-27 - j14)}{(-27 + j14)(-27 - j14)} = \frac{150 - j25}{27^2 + 14^2} = 0.1622 - j0.027 = 0.1644 \underline{/-9.46^\circ}$$

Method 2: (working in polar form):

$$z_3 = j = 1 \underline{/90^\circ}$$

$$z_4 = (3 - j4)^* = 5 \underline{/-53.13^\circ})^* = 5 \underline{/53.13^\circ}$$

$$z_5 = (-1 + j6) = \sqrt{37} \underline{/99.46^\circ}$$

$$z_6 = (2 + j)^2 = (\sqrt{5} \underline{/26.56^\circ})^2 = 5 \underline{/53.13^\circ}$$

Hence,

$$\begin{aligned} z_1 &= \frac{(1 \underline{/90^\circ})(5 \underline{/53.13^\circ})}{(\sqrt{37} \underline{/99.46^\circ})(5 \underline{/53.13^\circ})} \\ &= \frac{1}{\sqrt{37}} \underline{/90^\circ - 99.46^\circ} = 0.1644 \underline{/-9.46^\circ} \\ &= 0.1622 - j0.027 \end{aligned}$$

as obtained before.

(b) Let

$$z_2 = \left[\frac{z_7}{z_8}\right]^{1/2}$$

where

$$z_7 = 1 + j = \sqrt{2} \underline{/45^\circ}$$

and

$$z_8 = 4 - j8 = 4\sqrt{5} \underline{/-63.4^\circ}$$

Hence

$$\begin{aligned} \frac{z_7}{z_8} &= \frac{\sqrt{2} \underline{/45^\circ}}{4\sqrt{5} \underline{/-63.4^\circ}} = \frac{\sqrt{2}}{4\sqrt{5}} \underline{/45^\circ - -63.4^\circ} \\ &= 0.1581 \underline{/108.4^\circ} \end{aligned}$$

and

$$\begin{aligned} z_2 &= \sqrt{0.1581} \underline{/108.4^\circ/2} \\ &= 0.3976 \underline{/54.2^\circ} \end{aligned}$$

PRACTICE EXERCISE 9.5

Evaluate these complex numbers:

(a) $j^3\left[\frac{1 + j}{2 - j}\right]^2$

(b) $6 \underline{/30^\circ} + j5 - 3 + e^{j45^\circ}$

Answer: (a) $0.24 + j0.32$, (b) $2.903 + j8.707$.

EXAMPLE 9.6

Given that $\mathbf{A} = 10 \cos(10^8 t - 10x + 60°)\, \mathbf{a}_z$ and $\mathbf{B}_s = (20/j)\, \mathbf{a}_x + 10\, e^{j2\pi x/3}\, \mathbf{a}_y$, express $\mathbf{A}$ in phasor form and $\mathbf{B}_s$ in instantaneous form.

Solution:

$$\mathbf{A} = \text{Re}\,[10e^{j(\omega t - 10x + 60°)}\, \mathbf{a}_z]$$

where $\omega = 10^8$. Hence

$$\mathbf{A} = \text{Re}\,[10e^{j(60° - 10x)}\, \mathbf{a}_z\, e^{j\omega t}] = \text{Re}\,(\mathbf{A}_s e^{j\omega t})$$

or

$$\mathbf{A}_s = 10\, e^{j(60° - 10x)} \mathbf{a}_z$$

If

$$\begin{aligned}\mathbf{B}_s &= \frac{20}{j}\, \mathbf{a}_x + 10e^{j2\pi x/3}\mathbf{a}_y = -j20\mathbf{a}_x + 10e^{j2\pi x/3}\mathbf{a}_y \\ &= 20e^{-j\pi/2}\mathbf{a}_x + 10e^{j2\pi x/3}\mathbf{a}_y\end{aligned}$$

$$\begin{aligned}\mathbf{B} &= \text{Re}\,(\mathbf{B}_s e^{j\omega t}) \\ &= \text{Re}\,[20e^{j(\omega t - \pi/2)}\mathbf{a}_x + 10e^{j(\omega t + 2\pi x/3)}\mathbf{a}_y] \\ &= 20 \cos(\omega t - \pi/2)\mathbf{a}_x + 10 \cos\left(\omega t + \frac{2\pi x}{3}\right)\mathbf{a}_y \\ &= 20 \sin \omega t\, \mathbf{a}_x + 10 \cos\left(\omega t + \frac{2\pi x}{3}\right)\mathbf{a}_y\end{aligned}$$

PRACTICE EXERCISE 9.6

If $\mathbf{P} = 2 \sin(10t + x - \pi/4)\, \mathbf{a}_y$ and $\mathbf{Q}_s = e^{jx}(\mathbf{a}_x - \mathbf{a}_z) \sin \pi y$, determine the phasor form of $\mathbf{P}$ and the instantaneous form of $\mathbf{Q}_s$.

Answer: $2e^{j(x - 3\pi/4)}\mathbf{a}_y$, $\sin \pi y \cos(\omega t + x)(\mathbf{a}_x - \mathbf{a}_z)$.

EXAMPLE 9.7

The electric field and magnetic field in free space are given by

$$\mathbf{E} = \frac{50}{\rho} \cos(10^6 t + \beta z)\, \mathbf{a}_\phi \text{ V/m}$$

$$\mathbf{H} = \frac{H_o}{\rho} \cos(10^6 t + \beta z)\, \mathbf{a}_\rho \text{ A/m}$$

Express these in phasor form and determine the constants H_o and β such that the fields satisfy Maxwell's equations.

Solution:

The instantaneous forms of **E** and **H** are written as

$$\mathbf{E} = \text{Re}\,(\mathbf{E}_s e^{j\omega t}), \qquad \mathbf{H} = \text{Re}\,(\mathbf{H}_s e^{j\omega t}) \tag{9.7.1}$$

where $\omega = 10^6$ and phasors $\mathbf{E}_s$ and $\mathbf{H}_s$ are given by

$$\mathbf{E}_s = \frac{50}{\rho}\, e^{j\beta z}\mathbf{a}_\phi, \qquad \mathbf{H}_s = \frac{H_o}{\rho}\, e^{j\beta z}\mathbf{a}_\rho \tag{9.7.2}$$

For free space, $\rho_v = 0$, $\sigma = 0$, $\varepsilon = \varepsilon_o$, and $\mu = \mu_o$ so Maxwell's equations become

$$\nabla \cdot \mathbf{D} = \varepsilon_o \nabla \cdot \mathbf{E} = 0 \rightarrow \nabla \cdot \mathbf{E}_s = 0 \tag{9.7.3}$$

$$\nabla \cdot \mathbf{B} = \mu_o \nabla \cdot \mathbf{H} = 0 \rightarrow \nabla \cdot \mathbf{H}_s = 0 \tag{9.7.4}$$

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} \rightarrow \nabla \times \mathbf{H}_s = j\omega\varepsilon_o \mathbf{E}_s \tag{9.7.5}$$

$$\nabla \times \mathbf{E} = -\mu_o \frac{\partial \mathbf{H}}{\partial t} \quad \rightarrow \nabla \times \mathbf{E}_s = -j\omega\mu_o \mathbf{H}_s \tag{9.7.6}$$

Substituting eq. (9.7.2) into eqs. (9.7.3) and (9.7.4), it is readily verified that two Maxwell's equations are satisfied; that is,

$$\nabla \cdot \mathbf{E}_s = \frac{1}{\rho}\frac{\partial}{\partial \phi}(E_{\phi s}) = 0$$

$$\nabla \cdot \mathbf{H}_s = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_{\rho s}) = 0$$

Now

$$\nabla \times \mathbf{H}_s = \nabla \times \left(\frac{H_o}{\rho}\, e^{j\beta z}\, \mathbf{a}_\rho\right) = \frac{jH_o\beta}{\rho}\, e^{j\beta z}\, \mathbf{a}_\phi \tag{9.7.7}$$

Substituting eqs. (9.7.2) and (9.7.7) into eq. (9.7.5), we have

$$\frac{jH_o\beta}{\rho}\, e^{j\beta z}\mathbf{a}_\phi = j\omega\varepsilon_o \frac{50}{\rho}\, e^{j\beta z}\mathbf{a}_\phi$$

or

$$H_o\beta = 50\,\omega\varepsilon_o \tag{9.7.8}$$

Similarly, substituting eq. (9.7.2) into (9.7.6) gives

$$-j\beta\frac{50}{\rho}\, e^{j\beta z}\mathbf{a}_\rho = -j\omega\mu_o \frac{H_o}{\rho}\, e^{j\beta z}\mathbf{a}_\rho$$

or

$$\frac{H_o}{\beta} = \frac{50}{\omega\mu_o} \tag{9.7.9}$$

Multiplying eq. (9.7.8) with eq. (9.7.9) yields

$$H_o^2 = (50)^2 \frac{\varepsilon_o}{\mu_o}$$

or

$$H_o = \pm 50\sqrt{\varepsilon_o/\mu_o} = \pm \frac{50}{120\pi} = \pm 0.1326$$

Dividing eq. (9.7.8) by eq. (9.7.9), we get

$$\beta^2 = \omega^2 \mu_o \varepsilon_o$$

or

$$\beta = \pm \omega\sqrt{\mu_o\varepsilon_o} = \pm \frac{\omega}{c} = \pm \frac{10^6}{3 \times 10^8}$$
$$= \pm 3.33 \times 10^{-3}$$

In view of eq. (9.7.8), $H_o = 0.1326$, $\beta = 3.33 \times 10^{-3}$ or $H_o = -0.1326$, $\beta = -3.33 \times 10^{-3}$; only these will satisfy Maxwell's four equations.

PRACTICE EXERCISE 9.7

In air, $\mathbf{E} = \frac{\sin\theta}{r}\cos(6 \times 10^7 t - \beta r)\,\mathbf{a}_\phi$ V/m.

Find β and $\mathbf{H}$.

Answer: 0.2 rad/m, $-\frac{1}{12\pi r^2}\cos\theta\sin(6 \times 10^7 t - 0.2r)\,\mathbf{a}_r - \frac{1}{120\pi r}\sin\theta \times \cos(6 \times 10^7 t - 0.2r)\,\mathbf{a}_\theta$ A/m.

EXAMPLE 9.8

In a medium characterized by $\sigma = 0$, $\mu = \mu_o$, ε_o, and

$$\mathbf{E} = 20\sin(10^8 t - \beta z)\,\mathbf{a}_y \text{ V/m}$$

calculate β and $\mathbf{H}$.

Solution:

This problem can be solved directly in time domain or using phasors. As in the previous example, we find β and $\mathbf{H}$ by making $\mathbf{E}$ and $\mathbf{H}$ satisfy Maxwell's four equations.

Method 1 (time domain): Let us solve this problem the harder way—in time domain. It is evident that Gauss's law for electric fields is satisfied; that is,

$$\nabla \cdot \mathbf{E} = \frac{\partial E_y}{\partial y} = 0$$

From Faraday's law,

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \rightarrow \qquad \mathbf{H} = -\frac{1}{\mu} \int (\nabla \times \mathbf{E})\, dt$$

But

$$\nabla \times \mathbf{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -\frac{\partial E_y}{\partial z} \mathbf{a}_x + \frac{\partial E_y}{\partial x} \mathbf{a}_z$$
$$= 20\beta \cos (10^8 t - \beta z)\, \mathbf{a}_x + 0$$

Hence,

$$\mathbf{H} = -\frac{20\beta}{\mu} \int \cos (10^8 t - \beta z)\, dt\, \mathbf{a}_x$$
$$= -\frac{20\beta}{\mu 10^8} \sin (10^8 t - \beta z)\, \mathbf{a}_x \tag{9.8.1}$$

It is readily verified that

$$\nabla \cdot \mathbf{H} = \frac{\partial H_x}{\partial x} = 0$$

showing that Gauss's law for magnetic fields is satisfied. Lastly, from Ampere's law

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \qquad \rightarrow \qquad \mathbf{E} = \frac{1}{\varepsilon} \int (\nabla \times \mathbf{H})\, dt \tag{9.8.2}$$

because $\sigma = 0$.
But

$$\nabla \times \mathbf{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & 0 & 0 \end{vmatrix} = -\frac{\partial H_x}{\partial z} \mathbf{a}_y - \frac{\partial H_x}{\partial y} \mathbf{a}_z$$
$$= \frac{20\beta^2}{\mu 10^8} \cos (10^8 t - \beta z)\, \mathbf{a}_y + 0$$

where **H** in eq. (9.8.1) has been substituted. Thus eq. (9.8.2) becomes

$$\mathbf{E} = \frac{20\beta^2}{\mu\varepsilon 10^8} \int \cos (10^8 t - \beta z)\, dt\, \mathbf{a}_y$$
$$= \frac{20\beta^2}{\mu\varepsilon 10^{16}} \sin (10^8 t - \beta z)\, \mathbf{a}_y$$

Comparing this with the given **E**, we have

$$\frac{20\beta^2}{\mu\varepsilon 10^{16}} = 20$$

or

$$\beta = \pm 10^8\sqrt{\mu\varepsilon} = \pm 10^8\sqrt{\mu_o \cdot 4\varepsilon_o} = \pm\frac{10^8(2)}{c} = \pm\frac{10^8(2)}{3 \times 10^8}$$

$$= \pm\frac{2}{3}$$

From eq. (9.8.1),

$$\mathbf{H} = \pm\frac{20\,(2/3)}{4\pi \cdot 10^{-7}(10^8)} \sin\left(10^8 t \pm \frac{2z}{3}\right)\mathbf{a}_x$$

or

$$\mathbf{H} = \pm\frac{1}{3\pi}\sin\left(10^8 t \pm \frac{2z}{3}\right)\mathbf{a}_x \text{ A/m}$$

Method 2 (using phasors):

$$\mathbf{E} = \text{Im}\,(E_s e^{j\omega t}) \quad \rightarrow \quad \mathbf{E}_s = 20e^{-j\beta z}\,\mathbf{a}_y \tag{9.8.3}$$

where $\omega = 10^8$.
Again

$$\nabla \cdot \mathbf{E}_s = \frac{\partial E_{ys}}{\partial y} = 0$$

$$\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s \quad \rightarrow \quad \mathbf{H}_s = \frac{\nabla \times \mathbf{E}_s}{-j\omega\mu}$$

or

$$\mathbf{H}_s = \frac{1}{-j\omega\mu}\left[-\frac{\partial E_{ys}}{\partial z}\mathbf{a}_x\right] = -\frac{20\beta}{\omega\mu}e^{-j\beta z}\,\mathbf{a}_x \tag{9.8.4}$$

Notice that $\nabla \cdot \mathbf{H}_s = 0$ is satisfied.

$$\nabla \times \mathbf{H}_s = j\omega\varepsilon\mathbf{E}_s \quad \rightarrow \quad \mathbf{E}_s = \frac{\nabla \times \mathbf{H}_s}{j\omega\varepsilon} \tag{9.8.5}$$

Substituting $\mathbf{H}_s$ in eq. (9.8.4) into eq. (9.8.5) gives

$$\mathbf{E}_s = \frac{1}{j\omega\varepsilon}\frac{\partial H_{xs}}{\partial z}\mathbf{a}_y = \frac{20\beta^2 e^{-j\beta z}}{\omega^2\mu\varepsilon}\mathbf{a}_y$$

Comparing this with the given $\mathbf{E}_s$ in eq. (9.8.3), we have

$$20 = \frac{20\beta^2}{\omega^2\mu\varepsilon}$$

or

$$\beta = \pm \omega\sqrt{\mu\varepsilon} = \pm\frac{2}{3}$$

as obtained before. From eq. (9.8.4),

$$\mathbf{H}_s = \pm\frac{20(2/3)\,e^{\pm j\beta z}}{10^8(4\pi \times 10^{-7})}\,\mathbf{a}_x = \pm\frac{1}{3\pi}\,e^{\pm j\beta z}\,\mathbf{a}_x$$

$$\begin{aligned}\mathbf{H} &= \text{Im}\,(\mathbf{H}_s e^{j\omega t})\\ &= \pm\frac{1}{3\pi}\sin\,(10^8 t \pm \beta z)\,\mathbf{a}_x\ \text{A/m}\end{aligned}$$

as obtained before. It should be noticed that working with phasors provides a considerable simplification compared with working directly in time domain. Also, notice that we have used

$$\mathbf{A} = \text{Im}\,(\mathbf{A}_s e^{j\omega t})$$

because the given $\mathbf{E}$ is in sine form and not cosine. We could have used

$$\mathbf{A} = \text{Re}\,(\mathbf{A}_s e^{j\omega t})$$

in which case sine is expressed in terms of cosine and eq. (9.8.3) would be

$$\mathbf{E} = 20\cos\,(10^8 t - \beta z - 90°)\,\mathbf{a}_y = \text{Re}\,(\mathbf{E}_s e^{j\omega t})$$

or

$$\mathbf{E}_s = 20e^{-j\beta z - j90°}\mathbf{a}_y = -j20e^{-j\beta z}\mathbf{a}_y$$

and we follow the same procedure.

PRACTICE EXERCISE 9.8

A medium is characterized by $\sigma = 0$, $\mu = 2\mu_o$ and $\varepsilon = 5\varepsilon_o$. If $\mathbf{H} = 2\cos\,(\omega t - 3y)\,\mathbf{a}_z$ A/m, calculate ω and $\mathbf{E}$.

Answer: 2.846×10^8 rad/s, $-476.8\cos\,(2.846 \times 10^8 t - 3y)\,\mathbf{a}_x$ V/m.

SUMMARY

1. In this chapter, we have introduced two fundamental concepts: electromotive force (emf), based on Faraday's experiments, and displacement current, which resulted from Maxwell's hypothesis. These concepts call for modifications in Maxwell's curl equations obtained for static EM fields to accommodate the time dependence of the fields.
2. Faraday's law states that the induced emf is given by ($N = 1$)

$$V_{\text{emf}} = -\frac{\partial \Psi}{\partial t}$$

For transformer emf, $V_{\text{emf}} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$

and for motional emf, $V_{\text{emf}} = \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$.

3. The displacement current

$$I_d = \int \mathbf{J}_d \cdot d\mathbf{S}$$

where $\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}$ (displacement current density), is a modification to Ampere's circuit law. This modification attributed to Maxwell predicted electromagnetic waves several years before it was verified experimentally by Hertz.

4. In differential form, Maxwell's equations for dynamic fields are:

$$\nabla \cdot \mathbf{D} = \rho_v$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Each differential equation has its integral counterpart (see Tables 9.1 and 9.2) that can be derived from the differential form using Stokes's or divergence theorem. Any EM field must satisfy the four Maxwell's equations simultaneously.

5. Time-varying electric scalar potential $V(x, y, z, t)$ and magnetic vector potential $\mathbf{A}(x, y, z, t)$ are shown to satisfy wave equations if Lorentz's condition is assumed.

6. Time-harmonic fields are those that vary sinusoidally with time. They are easily expressed in phasors, which are more convenient to work with. Using the cosine reference, the instantaneous vector quantity $\mathbf{A}(x, y, z, t)$ is related to its phasor form $\mathbf{A}_s(x, y, z)$ according to

$$\mathbf{A}(x, y, z, t) = \text{Re}\,[\mathbf{A}_s(x, y, z)\, e^{j\omega t}]$$

EVIEW QUESTIONS

9.1 The flux through each turn of a 100-turn coil is $(t^3 - 2t)$ mWb, where t is in seconds. The induced emf at $t = 2$ s is

(a) 1 V
(b) −1 V
(c) 4 mV
(d) 0.4 V
(e) −0.4 V

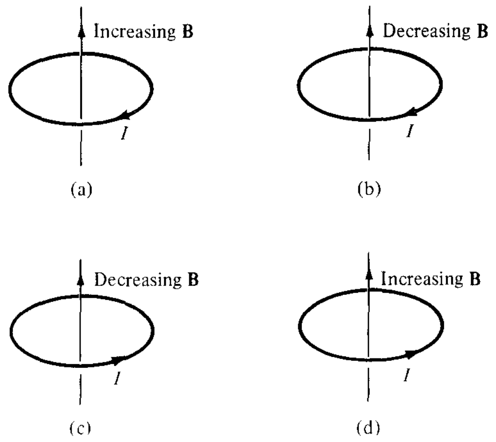

Figure 9.13 For Review Question 9.2.

9.2 Assuming that each loop is stationary and the time-varying magnetic field **B** induces current I, which of the configurations in Figure 9.13 are incorrect?

9.3 Two conducting coils 1 and 2 (identical except that 2 is split) are placed in a uniform magnetic field that decreases at a constant rate as in Figure 9.14. If the plane of the coils is perpendicular to the field lines, which of the following statements is true?

(a) An emf is induced in both coils.
(b) An emf is induced in split coil 2.
(c) Equal joule heating occurs in both coils.
(d) Joule heating does not occur in either coil.

9.4 A loop is rotating about the y-axis in a magnetic field $\mathbf{B} = B_o \sin \omega t \, \mathbf{a}_x$ Wb/m^2. The voltage induced in the loop is due to

(a) Motional emf
(b) Transformer emf
(c) A combination of motional and transformer emf
(d) None of the above

9.5 A rectangular loop is placed in the time-varying magnetic field $\mathbf{B} = 0.2 \cos 150\pi t \mathbf{a}_z$ Wb/m^2 as shown in Figure 9.15. V_1 is not equal to V_2.

(a) True (b) False

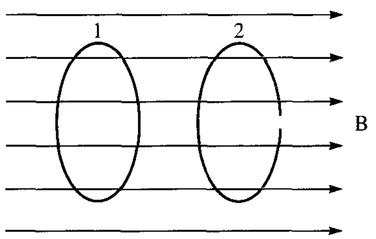

Figure 9.14 For Review Question 9.3.

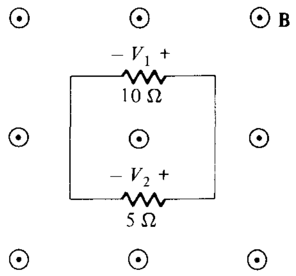

Figure 9.15 For Review Question 9.5 and Problem 9.10.

9.6 The concept of displacement current was a major contribution attributed to

(a) Faraday
(b) Lenz
(c) Maxwell
(d) Lorentz
(e) Your professor

9.7 Identify which of the following expressions are not Maxwell's equations for time-varying fields:

(a) $\nabla \cdot \mathbf{J} + \dfrac{\partial \rho_v}{\partial t} = 0$

(b) $\nabla \cdot \mathbf{D} = \rho_v$

(c) $\nabla \cdot \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$

(d) $\oint \mathbf{H} \cdot d\mathbf{l} = \int \left(\sigma \mathbf{E} + \varepsilon \dfrac{\partial \mathbf{E}}{\partial t}\right) \cdot d\mathbf{S}$

(e) $\oint \mathbf{B} \cdot d\mathbf{S} = 0$

9.8 An EM field is said to be nonexistent or not Maxwellian if it fails to satisfy Maxwell's equations and the wave equations derived from them. Which of the following fields in free space are not Maxwellian?

(a) $\mathbf{H} = \cos x \cos 10^6 t\, \mathbf{a}_x$

(b) $\mathbf{E} = 100 \cos \omega t\, \mathbf{a}_x$

(c) $\mathbf{D} = e^{-10y} \sin (10^5 - 10y)\, \mathbf{a}_z$

(d) $\mathbf{B} = 0.4 \sin 10^4 t\, \mathbf{a}_z$

(e) $\mathbf{H} = 10 \cos \left(10^5 t - \dfrac{z}{10}\right) \mathbf{a}_x$

(f) $\mathbf{E} = \dfrac{\sin \theta}{r} \cos (\omega t - r\omega\sqrt{\mu_o \varepsilon_o})\, \mathbf{a}_\theta$

(g) $\mathbf{B} = (1 - \rho^2) \sin \omega t\, \mathbf{a}_z$

9.9 Which of the following statements is not true of a phasor?

(a) It may be a scalar or a vector.
(b) It is a time-dependent quantity.
(c) A phasor V_s may be represented as $V_o \angle\theta$ or $V_o e^{j\theta}$ where $V_o = |V_s|$.
(d) It is a complex quantity.

9.10 If $\mathbf{E}_s = 10\, e^{j4x}\, \mathbf{a}_y$, which of these is not a correct representation of $\mathbf{E}$?

(a) Re $(\mathbf{E}_s e^{j\omega t})$
(b) Re $(\mathbf{E}_s e^{-j\omega t})$
(c) Im $(\mathbf{E}_s e^{j\omega t})$
(d) $10 \cos (\omega t + j4x)\, \mathbf{a}_y$
(e) $10 \sin (\omega t + 4x)\, \mathbf{a}_y$

Answers: 9.1b, 9.2b, d, 9.3a, 9.4c, 9.5a, 9.6c, 9.7a, b, d, g, 9.8b, 9.9a,c, 9.10d.

PROBLEMS

9.1 A conducting circular loop of radius 20 cm lies in the $z = 0$ plane in a magnetic field $\mathbf{B} = 10 \cos 377t\, \mathbf{a}_z$ mWb/m^2. Calculate the induced voltage in the loop.

9.2 A rod of length ℓ rotates about the z-axis with an angular velocity ω. If $\mathbf{B} = B_o \mathbf{a}_z$, calculate the voltage induced on the conductor.

9.3 A 30-cm by 40-cm rectangular loop rotates at 130 rad/s in a magnetic field 0.06 Wb/m^2 normal to the axis of rotation. If the loop has 50 turns, determine the induced voltage in the loop.

9.4 Figure 9.16 shows a conducting loop of area 20 cm^2 and resistance 4 Ω. If $\mathbf{B} = 40 \cos 10^4 t \mathbf{a}_z$ mWb/m^2, find the induced current in the loop and indicate its direction.

9.5 Find the induced emf in the V-shaped loop of Figure 9.17. (a) Take $\mathbf{B} = 0.1\mathbf{a}_z$ Wb/m^2 and $\mathbf{u} = 2\mathbf{a}_x$ m/s and assume that the sliding rod starts at the origin when $t = 0$. (b) Repeat part (a) if $\mathbf{B} = 0.5x\mathbf{a}_z$ Wb/m^2.

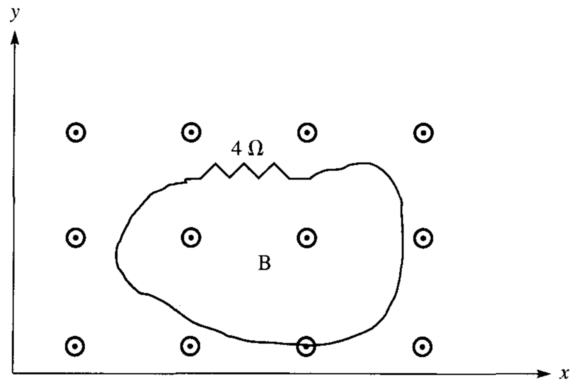

Figure 9.16 For Problem 9.4.

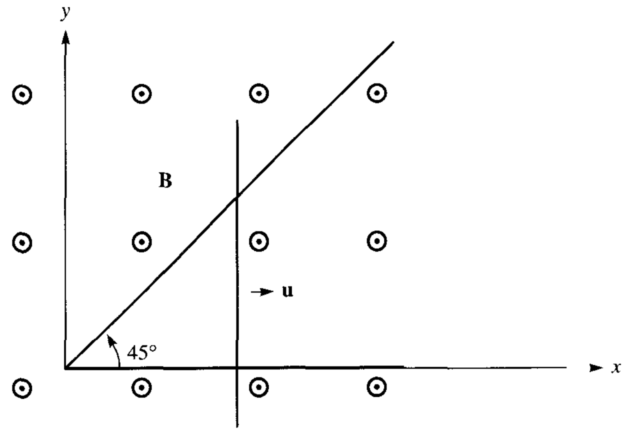

Figure 9.17 For Problem 9.5.

***9.6** A square loop of side a recedes with a uniform velocity $u_o \mathbf{a}_y$ from an infinitely long filament carrying current I along $\mathbf{a}_z$ as shown in Figure 9.18. Assuming that $\rho = \rho_o$ at time $t = 0$, show that the emf induced in the loop at $t > 0$ is

$$V_{emf} = \frac{u_o a^2 \mu_o I}{2\pi\rho(\rho + a)}$$

***9.7** A conducting rod moves with a constant velocity of $3\mathbf{a}_z$ m/s parallel to a long straight wire carrying current 15 A as in Figure 9.19. Calculate the emf induced in the rod and state which end is at higher potential.

***9.8** A conducting bar is connected via flexible leads to a pair of rails in a magnetic field $\mathbf{B} = 6 \cos 10t\, \mathbf{a}_x$ mWb/m^2 as in Figure 9.20. If the z-axis is the equilibrium position of the bar and its velocity is $2 \cos 10t\, \mathbf{a}_y$ m/s, find the voltage induced in it.

9.9 A car travels at 120 km/hr. If the earth's magnetic field is 4.3×10^{-5} Wb/m^2, find the induced voltage in the car bumper of length 1.6 m. Assume that the angle between the earth magnetic field and the normal to the car is 65°.

***9.10** If the area of the loop in Figure 9.15 is 10 cm^2, calculate V_1 and V_2.

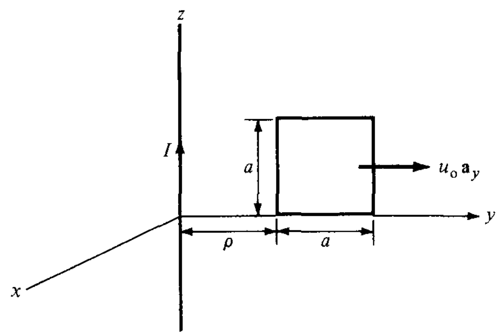

Figure 9.18 For Problem 9.6.

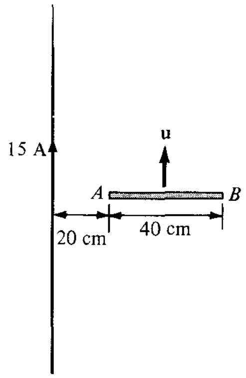

Figure 9.19 For Problem 9.7.

9.11 As portrayed in Figure 9.21, a bar magnet is thrust toward the center of a coil of 10 turns and resistance 15 Ω. If the magnetic flux through the coil changes from 0.45 Wb to 0.64 Wb in 0.02 s, what is the magnitude and direction (as viewed from the side near the magnet) of the induced current?

9.12 The cross section of a homopolar generator disk is shown in Figure 9.22. The disk has inner radius $\rho_1 = 2$ cm and outer radius $\rho_2 = 10$ cm and rotates in a uniform magnetic field 15 mWb/m^2 at a speed of 60 rad/s. Calculate the induced voltage.

9.13 A 50-V voltage generator at 20 MHz is connected to the plates of an air dielectric parallel-plate capacitor with plate area 2.8 cm^2 and separation distance 0.2 mm. Find the maximum value of displacement current density and displacement current.

9.14 The ratio J/J_d (conduction current density to displacement current density) is very important at high frequencies. Calculate the ratio at 1 GHz for:

(a) distilled water ($\mu = \mu_o$, $\varepsilon = 81\varepsilon_o$, $\sigma = 2 \times 10^{-3}$ S/m)

(b) sea water ($\mu = \mu_o$, $\varepsilon = 81\varepsilon_o$, $\sigma = 25$ S/m)

(c) limestone ($\mu = \mu_o$, $\varepsilon = 5\varepsilon_o$, $\sigma = 2 \times 10^{-4}$ S/m)

9.15 Assuming that sea water has $\mu = \mu_o$, $\varepsilon = 81\varepsilon_o$, $\sigma = 20$ S/m, determine the frequency at which the conduction current density is 10 times the displacement current density in magnitude.

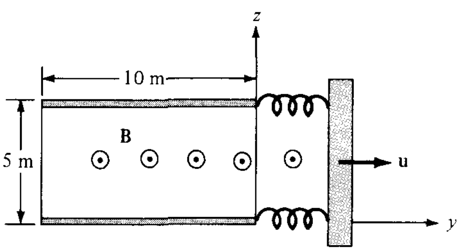

Figure 9.20 For Problem 9.8.

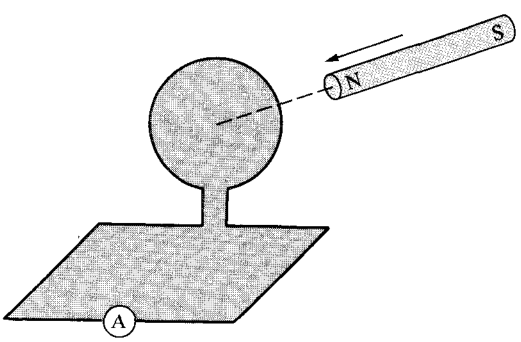

Figure 9.21 For Problem 9.11.

9.16 A conductor with cross-sectional area of 10 cm^2 carries a conduction current 0.2 sin $10^9 t$ mA. Given that $\sigma = 2.5 \times 10^6$ S/m and $\varepsilon_r = 6$, calculate the magnitude of the displacement current density.

9.17 (a) Write Maxwell's equations for a linear, homogeneous medium in terms of $\mathbf{E}_s$ and $\mathbf{H}_s$ only assuming the time factor $e^{-j\omega t}$.

(b) In Cartesian coordinates, write the point form of Maxwell's equations in Table 9.2 as eight scalar equations.

9.18 Show that in a source-free region ($\mathbf{J} = 0$, $\rho_v = 0$), Maxwell's equations can be reduced to two. Identify the two all-embracing equations.

9.19 In a linear homogeneous and isotropic conductor, show that the charge density ρ_v satisfies

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon} \rho_v = 0$$

9.20 Assuming a source-free region, derive the diffusion equation

$$\nabla^2 \mathbf{E} = \mu\sigma \frac{\partial \mathbf{E}}{\partial t}$$

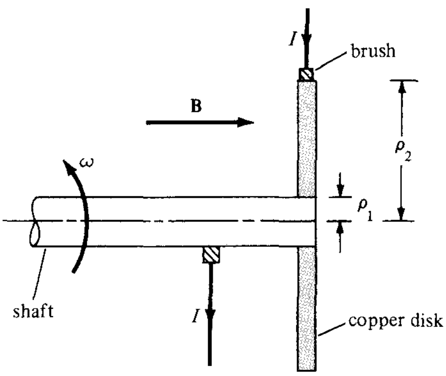

Figure 9.22 For Problem 9.12.

9.21 In a certain region,

$$\mathbf{J} = (2y\mathbf{a}_x + xz\mathbf{a}_y + z^3\mathbf{a}_z) \sin 10^4 t \text{ A/m}$$

find ρ_v if $\rho_v(x, y, 0, t) = 0$.

9.22 In a charge-free region for which $\sigma = 0$, $\varepsilon = \varepsilon_o\varepsilon_r$, and $\mu = \mu_o$,

$$\mathbf{H} = 5 \cos(10^{11}t - 4y)\mathbf{a}_z \text{ A/m}$$

find: (a) $\mathbf{J}_d$ and $\mathbf{D}$, (b) ε_r.

9.23 In a certain region with $\sigma = 0$, $\mu = \mu_o$, and $\varepsilon = 6.25\varepsilon_o$, the magnetic field of an EM wave is

$$\mathbf{H} = 0.6 \cos \beta x \cos 10^8 t \, \mathbf{a}_z \text{ A/m}$$

Find β and the corresponding $\mathbf{E}$ using Maxwell's equations.

***9.24** In a nonmagnetic medium,

$$\mathbf{E} = 50 \cos(10^9 t - 8x)\mathbf{a}_y + 40 \sin(10^9 t - 8x)\mathbf{a}_z \text{ V/m}$$

find the dielectric constant ε_r and the corresponding $\mathbf{H}$.

9.25 Check whether the following fields are genuine EM fields, i.e., they satisfy Maxwell's equations. Assume that the fields exist in charge-free regions.

(a) $\mathbf{A} = 40 \sin(\omega t + 10x)\mathbf{a}_z$

(b) $\mathbf{B} = \dfrac{10}{\rho} \cos(\omega t - 2\rho)\mathbf{a}_\phi$

(c) $\mathbf{C} = \left(3\rho^2 \cot \phi \mathbf{a}_\rho + \dfrac{\cos \phi}{\rho} \mathbf{a}_\phi\right) \sin \omega t$

(d) $\mathbf{D} = \dfrac{1}{r} \sin \theta \sin(\omega t - 5r)\mathbf{a}_\theta$

****9.26** Given the total electromagnetic energy

$$W = \frac{1}{2} \int (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) \, dv$$

show from Maxwell's equations that

$$\frac{\partial W}{\partial t} = -\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} - \int_v \mathbf{E} \cdot \mathbf{J} \, dv$$

9.27 In free space,

$$\mathbf{H} = \rho(\sin \phi \mathbf{a}_\rho + 2 \cos \phi \mathbf{a}_\phi) \cos 4 \times 10^6 t \text{ A/m}$$

find $\mathbf{J}_d$ and $\mathbf{E}$.

9.28 An antenna radiates in free space and

$$\mathbf{H} = \frac{12 \sin \theta}{r} \cos(2\pi \times 10^8 t - \beta r)\mathbf{a}_\theta \text{ mA/m}$$

find the corresponding $\mathbf{E}$ in terms of β.

***9.29** The electric field in air is given by $\mathbf{E} = \rho t e^{-\rho - t}\mathbf{a}_\phi$ V/m; find $\mathbf{B}$ and $\mathbf{J}$.

****9.30** In free space ($\rho_v = 0, \mathbf{J} = 0$). Show that

$$\mathbf{A} = \frac{\mu_0}{4\pi r}(\cos \theta\, \mathbf{a}_r - \sin \theta\, \mathbf{a}_\theta)e^{j\omega(t - r/c)}$$

satisfies the wave equation in eq. (9.52). Find the corresponding V. Take c as the speed of light in free space.

9.31 Evaluate the following complex numbers and express your answers in polar form:

(a) $(4\,\angle 30^\circ - 10\,\angle 50^\circ)^{1/2}$

(b) $\dfrac{1 + j2}{6 + j8 - 7\,\angle 15^\circ}$

(c) $\dfrac{(3 + j4)^2}{12 - j7 + (-6 + j10)^*}$

(d) $\dfrac{(3.6\,\angle -200^\circ)^{1/2}}{(2.4\,\angle 45^\circ)^2(-5 + j8)^*}$

9.32 Write the following time-harmonic fields as phasors:

(a) $\mathbf{E} = 4\cos(\omega t - 3x - 10^\circ)\,\mathbf{a}_y - \sin(\omega t + 3x + 20^\circ)\,\mathbf{a}_z$

(b) $\mathbf{H} = \dfrac{\sin \theta}{r}\cos(\omega t - 5r)\mathbf{a}_\theta$

(c) $\mathbf{J} = 6e^{-3x}\sin(\omega t - 2x)\mathbf{a}_y + 10e^{-x}\cos(\omega t - 5x)\mathbf{a}_z$

9.33 Express the following phasors in their instantaneous forms:

(a) $\mathbf{A}_s = (4 - 3j)e^{-j\beta x}\mathbf{a}_y$

(b) $\mathbf{B}_s = \dfrac{20}{\rho}e^{-j2z}\mathbf{a}_\rho$

(c) $\mathbf{C}_s = \dfrac{10}{r^2}(1 + j2)e^{-j\phi}\sin \theta\mathbf{a}_\phi$

9.34 Given $\mathbf{A} = 4\sin \omega t\mathbf{a}_x + 3\cos \omega t\mathbf{a}_y$ and $\mathbf{B}_s = j10ze^{-jz}\mathbf{a}_x$, express $\mathbf{A}$ in phase form and $\mathbf{B}_s$ in instantaneous form.

9.35 Show that in a linear homogeneous, isotropic source-free region, both $\mathbf{E}_s$ and $\mathbf{H}_s$ must satisfy the wave equation

$$\nabla^2\mathbf{A}_s + \gamma^2\mathbf{A}_s = 0$$

where $\gamma^2 = \omega^2\mu\varepsilon - j\omega\mu\sigma$ and $\mathbf{A}_s = \mathbf{E}_s$ or $\mathbf{H}_s$.

Chapter 10

ELECTROMAGNETIC WAVE PROPAGATION

How far you go in life depends on your being tender with the young, compassionate with the aged, sympathetic with the striving, and tolerant of the weak and the strong. Because someday in life you will have been all of these.

—GEORGE W. CARVER

10.1 INTRODUCTION

Our first application of Maxwell's equations will be in relation to electromagnetic wave propagation. The existence of EM waves, predicted by Maxwell's equations, was first investigated by Heinrich Hertz. After several calculations and experiments Hertz succeeded in generating and detecting radio waves, which are sometimes called Hertzian waves in his honor.

In general, **waves** are means of transporting energy or information.

Typical examples of EM waves include radio waves, TV signals, radar beams, and light rays. All forms of EM energy share three fundamental characteristics: they all travel at high velocity; in traveling, they assume the properties of waves; and they radiate outward from a source, without benefit of any discernible physical vehicles. The problem of radiation will be addressed in Chapter 13.

In this chapter, our major goal is to solve Maxwell's equations and derive EM wave motion in the following media:

1. Free space ($\sigma = 0, \varepsilon = \varepsilon_o, \mu = \mu_o$)
2. Lossless dielectrics ($\sigma = 0, \varepsilon = \varepsilon_r\varepsilon_o, \mu = \mu_r\mu_o$, or $\sigma \ll \omega\varepsilon$)
3. Lossy dielectrics ($\sigma \neq 0, \varepsilon = \varepsilon_r\varepsilon_o, \mu = \mu_r\mu_o$)
4. Good conductors ($\sigma \simeq \infty, \varepsilon = \varepsilon_o, \mu = \mu_r\mu_o$, or $\sigma \gg \omega\varepsilon$)

where ω is the angular frequency of the wave. Case 3, for lossy dielectrics, is the most general case and will be considered first. Once this general case is solved, we simply derive other cases (1, 2, and 4) from it as special cases by changing the values of σ, ε, and μ. However, before we consider wave motion in those different media, it is appropriate that we study the characteristics of waves in general. This is important for proper understanding of EM waves. The reader who is conversant with the concept of waves may skip Section 10.2. Power considerations, reflection, and transmission between two different media will be discussed later in the chapter.

10.2 WAVES IN GENERAL

A clear understanding of EM wave propagation depends on a grasp of what waves are in general.

A **wave** is a function of both space and time.

Wave motion occurs when a disturbance at point A, at time t_o, is related to what happens at point B, at time $t > t_o$. A wave equation, as exemplified by eqs. (9.51) and (9.52), is a partial differential equation of the second order. In one dimension, a scalar wave equation takes the form of

$$\frac{\partial^2 E}{\partial t^2} - u^2 \frac{\partial^2 E}{\partial z^2} = 0 \tag{10.1}$$

where u is the *wave velocity*. Equation (10.1) is a special case of eq. (9.51) in which the medium is source free ($\rho_v = 0, \mathbf{J} = 0$). It can be solved by following procedure, similar to that in Example 6.5. Its solutions are of the form

$$E^- = f(z - ut) \tag{10.2a}$$

$$E^+ = g(z + ut) \tag{10.2b}$$

or

$$E = f(z - ut) + g(z + ut) \tag{10.2c}$$

where f and g denote any function of $z - ut$ and $z + ut$, respectively. Examples of such functions include $z \pm ut$, $\sin k(z \pm ut)$, $\cos k(z \pm ut)$, and $e^{jk(z \pm ut)}$, where k is a constant. It can easily be shown that these functions all satisfy eq. (10.1).

If we particularly assume harmonic (or sinusoidal) time dependence $e^{j\omega t}$, eq. (10.1) becomes

$$\frac{d^2 E_s}{dz^2} + \beta^2 E_s = 0 \tag{10.3}$$

where $\beta = \omega/u$ and E_s is the phasor form of E. The solution to eq. (10.3) is similar to Case 3 of Example 6.5 [see eq. (6.5.12)]. With the time factor inserted, the possible solutions to eq. (10.3) are

$$E^+ = Ae^{j(\omega t - \beta z)} \tag{10.4a}$$

$$E^- = Be^{j(\omega t + \beta z)} \tag{10.4b}$$

and

$$E = Ae^{j(\omega t - \beta z)} + Be^{j(\omega t + \beta z)} \tag{10.4c}$$

where A and B are real constants.

For the moment, let us consider the solution in eq. (10.4a). Taking the imaginary part of this equation, we have

$$E = A \sin (\omega t - \beta z) \tag{10.5}$$

This is a sine wave chosen for simplicity; a cosine wave would have resulted had we taken the real part of eq. (10.4a). Note the following characteristics of the wave in eq. (10.5):

1. It is time harmonic because we assumed time dependence $e^{j\omega t}$ to arrive at eq. (10.5).
2. A is called the *amplitude* of the wave and has the same units as E.
3. $(\omega t - \beta z)$ is the *phase* (in radians) of the wave; it depends on time t and space variable z.
4. ω is the *angular frequency* (in radians/second); β is the *phase constant* or *wave number* (in radians/meter).

Due to the variation of E with both time t and space variable z, we may plot E as a function of t by keeping z constant and vice versa. The plots of $E(z, t = \text{constant})$ and $E(t, z = \text{constant})$ are shown in Figure 10.1(a) and (b), respectively. From Figure 10.1(a), we observe that the wave takes distance λ to repeat itself and hence λ is called the *wavelength* (in meters). From Figure 10.1(b), the wave takes time T to repeat itself; consequently T is known as the *period* (in seconds). Since it takes time T for the wave to travel distance λ at the speed u, we expect

$$\lambda = uT \tag{10.6a}$$

But $T = 1/f$, where f is the *frequency* (the number of cycles per second) of the wave in Hertz (Hz). Hence,

$$\boxed{u = f\lambda} \tag{10.6b}$$

Because of this fixed relationship between wavelength and frequency, one can identify the position of a radio station within its band by either the frequency or the wavelength. Usually the frequency is preferred. Also, because

$$\omega = 2\pi f \tag{10.7a}$$

$$\beta = \frac{\omega}{u} \tag{10.7b}$$

and

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \tag{10.7c}$$

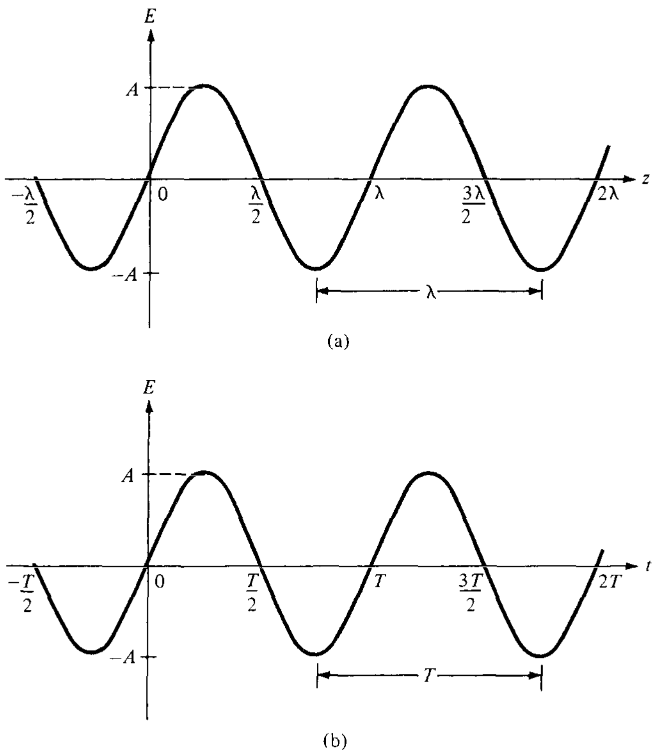

Figure 10.1 Plot of $E(z, t) = A \sin(\omega t - \beta z)$: **(a)** with constant t, **(b)** with constant z.

we expect from eqs. (10.6) and (10.7) that

$$\boxed{\beta = \frac{2\pi}{\lambda}} \tag{10.8}$$

Equation (10.8) shows that for every wavelength of distance traveled, a wave undergoes a phase change of 2π radians.

We will now show that the wave represented by eq. (10.5) is traveling with a velocity u in the $+z$ direction. To do this, we consider a fixed point P on the wave. We sketch eq. (10.5) at times $t = 0$, $T/4$, and $T/2$ as in Figure 10.2. From the figure, it is evident that as the wave advances with time, point P moves along $+z$ direction. Point P is a point of constant phase, therefore

$$\omega t - \beta z = \text{constant}$$

or

$$\frac{dz}{dt} = \frac{\omega}{\beta} = u \tag{10.9}$$

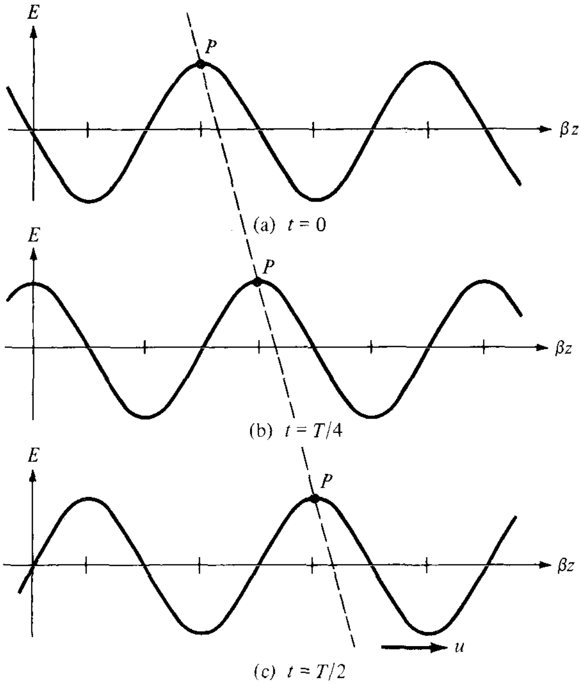

Figure 10.2 Plot of $E(z, t) = A \sin(\omega t - \beta z)$ at time **(a)** $t = 0$, **(b)** $t = T/4$, **(c)** $t = T/2$; P moves along $+z$ direction with velocity u.

which is the same as eq. (10.7b). Equation (10.9) shows that the wave travels with velocity u in the $+z$ direction. Similarly, it can be shown that the wave $B \sin(\omega t + \beta z)$ in eq. (10.4b) is traveling with velocity u in the $-z$ direction.

In summary, we note the following:

1. A wave is a function of both time and space.
2. Though time $t = 0$ is arbitrarily selected as a reference for the wave, a wave is without beginning or end.
3. A negative sign in $(\omega t \pm \beta z)$ is associated with a wave propagating in the $+z$ direction (forward traveling or positive-going wave) whereas a positive sign indicates that a wave is traveling in the $-z$ direction (backward traveling or negative-going wave).
4. Since $\sin(-\psi) = -\sin\psi = \sin(\psi \pm \pi)$, whereas $\cos(-\psi) = \cos\psi$,

$$\sin(\psi \pm \pi/2) = \pm\cos\psi \tag{10.10a}$$

$$\sin(\psi \pm \pi) = -\sin\psi \tag{10.10b}$$

$$\cos(\psi \pm \pi/2) = \mp\sin\psi \tag{10.10c}$$

$$\cos(\psi \pm \pi) = -\cos\psi \tag{10.10d}$$

where $\psi = \omega t \pm \beta z$. With eq. (10.10), any time-harmonic wave can be represented in the form of sine or cosine.

TABLE 10.1 Electromagnetic Spectrum

EM Phenomena	Examples of Uses	Approximate Frequency Range
Cosmic rays	Physics, astronomy	10^{14} GHz and above
Gamma rays	Cancer therapy	10^{10}–10^{13} GHz
X-rays	X-ray examination	10^{8}–10^{9} GHz
Ultraviolet radiation	Sterilization	10^{6}–10^{8} GHz
Visible light	Human vision	10^{5}–10^{6} GHz
Infrared radiation	Photography	10^{3}–10^{4} GHz
Microwave waves	Radar, microwave relays, satellite communication	3–300 GHz
Radio waves	UHF television	470–806 MHz
	VHF television, FM radio	54–216 MHz
	Short-wave radio	3–26 MHz
	AM radio	535–1605 kHz

A large number of frequencies visualized in numerical order constitute a *spectrum.* Table 10.1 shows at what frequencies various types of energy in the EM spectrum occur. Frequencies usable for radio communication occur near the lower end of the EM spectrum. As frequency increases, the manifestation of EM energy becomes dangerous to human beings.[1] Microwave ovens, for example, can pose a hazard if not properly shielded. The practical difficulties of using EM energy for communication purposes also increase as frequency increases, until finally it can no longer be used. As communication methods improve, the limit to usable frequency has been pushed higher. Today communication satellites use frequencies near 14 GHz. This is still far below light frequencies, but in the enclosed environment of fiber optics, light itself can be used for radio communication.[2]

EXAMPLE 10.1

The electric field in free space is given by

$$\mathbf{E} = 50 \cos (10^8 t + \beta x)\, \mathbf{a}_y \text{ V/m}$$

(a) Find the direction of wave propagation.

(b) Calculate β and the time it takes to travel a distance of $\lambda/2$.

(c) Sketch the wave at $t = 0$, $T/4$, and $T/2$.

Solution:

(a) From the positive sign in $(\omega t + \beta x)$, we infer that the wave is propagating along $-\mathbf{a}_x$. This will be confirmed in part (c) of this example.

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[1]See March 1987 special issue of *IEEE Engineering in Medicine and Biology Magazine* on "Effects of EM Radiation."

[2]See October 1980 issue of *IEEE Proceedings* on "Optical-Fiber Communications."

(b) In free space, $u = c$.

$$\beta = \frac{\omega}{c} = \frac{10^8}{3 \times 10^8} = \frac{1}{3}$$

or

$$\beta = 0.3333 \text{ rad/m}$$

If T is the period of the wave, it takes T seconds to travel a distance λ at speed c. Hence to travel a distance of $\lambda/2$ will take

$$t_1 = \frac{T}{2} = \frac{1}{2}\frac{2\pi}{\omega} = \frac{\pi}{10^8} = 31.42 \text{ ns}$$

Alternatively, because the wave is traveling at the speed of light c,

$$\frac{\lambda}{2} = ct_1 \qquad \text{or} \qquad t_1 = \frac{\lambda}{2c}$$

But

$$\lambda = \frac{2\pi}{\beta} = 6\pi$$

Hence,

$$t_1 = \frac{6\pi}{2(3 \times 10^8)} = 31.42 \text{ ns}$$

as obtained before.

(c) At $t = 0, \quad E_y = 50 \cos \beta x$

At $t = T/4, E_y = 50 \cos\left(\omega \cdot \frac{2\pi}{4\omega} + \beta x\right) = 50 \cos(\beta x + \pi/2)$
$= -50 \sin \beta x$

At $t = T/2, E_y = 50 \cos\left(\omega \cdot \frac{2\pi}{2\omega} + \beta x\right) = 50 \cos(\beta x + \pi)$
$= -50 \cos \beta x$

E_y at $t = 0, T/4, T/2$ is plotted against x as shown in Figure 10.3. Notice that a point P (arbitrarily selected) on the wave moves along $-\mathbf{a}_x$ as t increases with time. This shows that the wave travels along $-\mathbf{a}_x$.

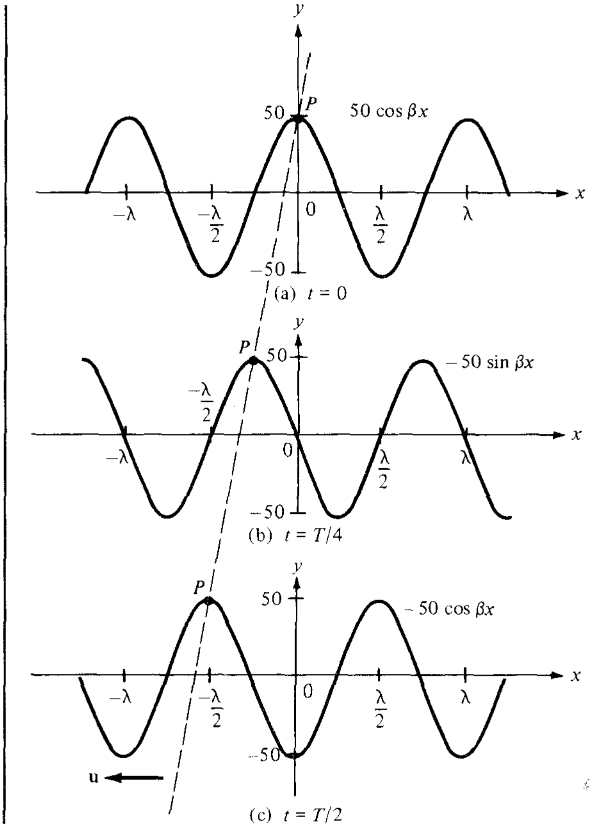

Figure 10.3 For Example 10.1; wave travels along $-\mathbf{a}_x$.

PRACTICE EXERCISE 10.1

In free space, $\mathbf{H} = 0.1 \cos(2 \times 10^8 t - kx)\,\mathbf{a}_y$ A/m. Calculate

(a) k, λ, and T

(b) The time t_1 it takes the wave to travel $\lambda/8$

(c) Sketch the wave at time t_1.

Answer: (a) 0.667 rad/m, 9.425 m, 31.42 ns, (b) 3.927 ns, (c) see Figure 10.4.

10.3 WAVE PROPAGATION IN LOSSY DIELECTRICS

As mentioned in Section 10.1, wave propagation in lossy dielectrics is a general case from which wave propagation in other types of media can be derived as special cases. Therefore, this section is foundational to the next three sections.

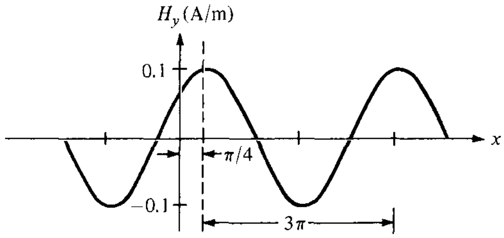

Figure 10.4 For Practice Exercise 10.1(c).

A **lossy dielectric** is a medium in which an EM wave loses power as it propagates due to poor conduction.

In other words, a lossy dielectric is a partially conducting medium (imperfect dielectric or imperfect conductor) with $\sigma \neq 0$, as distinct from a lossless dielectric (perfect or good dielectric) in which $\sigma = 0$.

Consider a linear, isotropic, homogeneous, lossy dielectric medium that is charge free ($\rho_v = 0$). Assuming and suppressing the time factor $e^{j\omega t}$, Maxwell's equations (see Table 9.2) become

$$\nabla \cdot \mathbf{E}_s = 0 \tag{10.11}$$

$$\nabla \cdot \mathbf{H}_s = 0 \tag{10.12}$$

$$\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s \tag{10.13}$$

$$\nabla \times \mathbf{H}_s = (\sigma + j\omega\varepsilon)\mathbf{E}_s \tag{10.14}$$

Taking the curl of both sides of eq. (10.13) gives

$$\nabla \times \nabla \times \mathbf{E}_s = -j\omega\mu\, \nabla \times \mathbf{H}_s \tag{10.15}$$

Applying the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{10.16}$$

to the left-hand side of eq. (10.15) and invoking eqs. (10.11) and (10.14), we obtain

$$\nabla(\underbrace{\nabla \cdot \mathbf{E}_s}_{0}) - \nabla^2\mathbf{E}_s = -j\omega\mu(\sigma + j\omega\varepsilon)\mathbf{E}_s$$

or

$$\boxed{\nabla^2\mathbf{E}_s - \gamma^2\mathbf{E}_s = 0} \tag{10.17}$$

where

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) \tag{10.18}$$

and γ is called the *propagation constant* (in per meter) of the medium. By a similar procedure, it can be shown that for the **H** field,

$$\nabla^2 \mathbf{H}_s - \gamma^2 \mathbf{H}_s = 0 \tag{10.19}$$

Equations (10.17) and (10.19) are known as homogeneous vector *Helmholtz's equations* or simply vector *wave equations.* In Cartesian coordinates, eq. (10.17), for example, is equivalent to three scalar wave equations, one for each component of **E** along $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$.

Since γ in eqs. (10.17) to (10.19) is a complex quantity, we may let

$$\boxed{\gamma = \alpha + j\beta} \tag{10.20}$$

We obtain α and β from eqs. (10.18) and (10.20) by noting that

$$-\text{Re}\,\gamma^2 = \beta^2 - \alpha^2 = \omega^2 \mu \varepsilon \tag{10.21}$$

and

$$|\gamma^2| = \beta^2 + \alpha^2 = \omega\mu \sqrt{\sigma^2 + \omega^2 \varepsilon^2} \tag{10.22}$$

From eqs. (10.21) and (10.22), we obtain

$$\alpha = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]} \tag{10.23}$$

$$\beta = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]} \tag{10.24}$$

Without loss of generality, if we assume that the wave propagates along $+\mathbf{a}_z$ and that $\mathbf{E}_s$ has only an x-component, then

$$\mathbf{E}_s = E_{xs}(z)\mathbf{a}_x \tag{10.25}$$

Substituting this into eq. (10.17) yields

$$(\nabla^2 - \gamma^2)E_{xs}(z) \tag{10.26}$$

Hence

$$\underbrace{\frac{\partial^2 E_{xs}(z)}{\partial x^2}}_{0} + \underbrace{\frac{\partial^2 E_{xs}(z)}{\partial y^2}}_{0} + \frac{\partial^2 E_{xs}(z)}{\partial z^2} - \gamma^2 E_{xs}(z) = 0$$

or

$$\left[\frac{d^2}{dz^2} - \gamma^2\right] E_{xs}(z) = 0 \tag{10.27}$$

This is a scalar wave equation, a linear homogeneous differential equation, with solution (see Case 2 in Example 6.5)

$$E_{xs}(z) = E_o e^{-\gamma z} + E'_o e^{\gamma z} \tag{10.28}$$

where E_o and E'_o are constants. The fact that the field must be finite at infinity requires that $E'_o = 0$. Alternatively, because $e^{\gamma z}$ denotes a wave traveling along $-\mathbf{a}_z$ whereas we assume wave propagation along $\mathbf{a}_z$, $E'_o = 0$. Whichever way we look at it, $E'_o = 0$. Inserting the time factor $e^{j\omega t}$ into eq. (10.28) and using eq. (10.20), we obtain

$$\mathbf{E}(z, t) = \text{Re}\,[E_{xs}(z)e^{j\omega t}\mathbf{a}_x] = \text{Re}\,(E_o e^{-\alpha z} e^{j(\omega t - \beta z)}\mathbf{a}_x)$$

or

$$\boxed{\mathbf{E}(z, t) = E_o e^{-\alpha z}\cos(\omega t - \beta z)\mathbf{a}_x} \tag{10.29}$$

A sketch of $|\mathbf{E}|$ at times $t = 0$ and $t = \Delta t$ is portrayed in Figure 10.5, where it is evident that $\mathbf{E}$ has only an x-component and it is traveling along the $+z$-direction. Having obtained $\mathbf{E}(z, t)$, we obtain $\mathbf{H}(z, t)$ either by taking similar steps to solve eq. (10.19) or by using eq. (10.29) in conjunction with Maxwell's equations as we did in Example 9.8. We will eventually arrive at

$$\mathbf{H}(z, t) = \text{Re}\,(H_o e^{-\alpha z} e^{j(\omega t - \beta z)}\,\mathbf{a}_y) \tag{10.30}$$

where

$$H_o = \frac{E_o}{\eta} \tag{10.31}$$

and η is a complex quantity known as the *intrinsic impedance* (in ohms) of the medium. It can be shown by following the steps taken in Example 9.8 that

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta|\underline{/\theta_\eta} = |\eta|e^{j\theta_\eta} \tag{10.32}$$

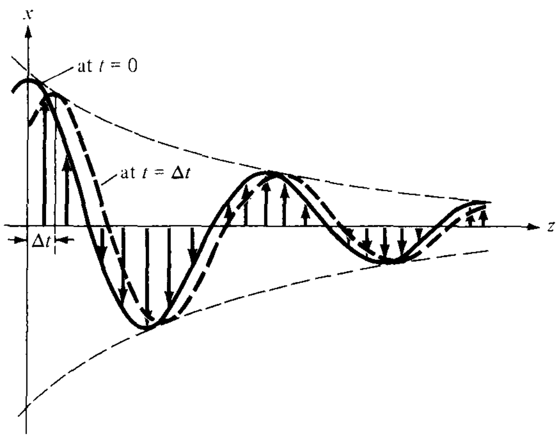

Figure 10.5 E-field with x-component traveling along $+z$-direction at times $t = 0$ and $t = \Delta t$; arrows indicate instantaneous values of E.

with

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}, \qquad \tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} \tag{10.33}$$

where $0 \leq \theta_\eta \leq 45°$. Substituting eqs. (10.31) and (10.32) into eq. (10.30) gives

$$\mathbf{H} = \text{Re}\left[\frac{E_o}{|\eta| e^{j\theta_\eta}} e^{-\alpha z} e^{j(\omega t - \beta z)} \mathbf{a}_y\right]$$

or

$$\mathbf{H} = \frac{E_o}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\, \mathbf{a}_y \tag{10.34}$$

Notice from eqs. (10.29) and (10.34) that as the wave propagates along $\mathbf{a}_z$, it decreases or attenuates in amplitude by a factor $e^{-\alpha z}$, and hence α is known as the *attenuation constant* or *attenuation factor* of the medium. It is a measure of the spatial rate of decay of the wave in the medium, measured in nepers per meter (Np/m) or in decibels per meter (dB/m). An attenuation of 1 neper denotes a reduction to e^{-1} of the original value whereas an increase of 1 neper indicates an increase by a factor of e. Hence, for voltages

$$1 \text{ Np} = 20 \log_{10} e = 8.686 \text{ dB} \tag{10.35}$$

From eq. (10.23), we notice that if $\sigma = 0$, as is the case for a lossless medium and free space, $\alpha = 0$ and the wave is not attenuated as it propagates. The quantity β is a measure of the phase shift per length and is called the *phase constant* or *wave number*. In terms of β, the wave velocity u and wavelength λ are, respectively, given by [see eqs. (10.7b) and (10.8)]

$$u = \frac{\omega}{\beta}, \qquad \lambda = \frac{2\pi}{\beta} \tag{10.36}$$

We also notice from eqs. (10.29) and (10.34) that $\mathbf{E}$ and $\mathbf{H}$ are out of phase by θ_η at any instant of time due to the complex intrinsic impedance of the medium. Thus at any time, $\mathbf{E}$ leads $\mathbf{H}$ (or $\mathbf{H}$ lags $\mathbf{E}$) by θ_η. Finally, we notice that the ratio of the magnitude of the conduction current density $\mathbf{J}$ to that of the displacement current density $\mathbf{J}_d$ in a lossy medium is

$$\frac{|\mathbf{J}_s|}{|\mathbf{J}_{ds}|} = \frac{|\sigma \mathbf{E}_s|}{|j\omega\varepsilon \mathbf{E}_s|} = \frac{\sigma}{\omega\varepsilon} = \tan\theta$$

or

$$\tan\theta = \frac{\sigma}{\omega\varepsilon} \tag{10.37}$$

where $\tan\theta$ is known as the *loss tangent* and θ is the *loss angle* of the medium as illustrated in Figure 10.6. Although a line of demarcation between good conductors and lossy dielectrics is not easy to make, $\tan\theta$ or θ may be used to determine how lossy a medium is. A medium is said to be a good (lossless or perfect) dielectric if $\tan\theta$ is very small ($\sigma \ll \omega\varepsilon$) or a good conductor if $\tan\theta$ is very large ($\sigma \gg \omega\varepsilon$). From the viewpoint of wave propagation, the characteristic behavior of a medium depends not only on its constitutive parameters σ, ε, and μ but also on the frequency of operation. A medium that is regarded as a good conductor at low frequencies may be a good dielectric at high frequencies. Note from eqs. (10.33) and (10.37) that

$$\theta = 2\theta_\eta \tag{10.38}$$

From eq. (10.14)

$$\begin{aligned}\nabla \times \mathbf{H}_s &= (\sigma + j\omega\varepsilon)\mathbf{E}_s = j\omega\varepsilon\left[1 - \frac{j\sigma}{\omega\varepsilon}\right]\mathbf{E}_s \\ &= j\omega\varepsilon_c\mathbf{E}_s\end{aligned} \tag{10.39}$$

where

$$\boxed{\varepsilon_c = \varepsilon\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]} \tag{10.40a}$$

or

$$\varepsilon_c = \varepsilon' - j\varepsilon'' \tag{10.40b}$$

and $\varepsilon' = \varepsilon$, $\varepsilon'' = \sigma/\omega$; ε_c is called the *complex permittivity* of the medium. We observe that the ratio of ε'' to ε' is the loss tangent of the medium; that is,

$$\tan\theta = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon} \tag{10.41}$$

In subsequent sections, we will consider wave propagation in other types of media. which may be regarded as special cases of what we have considered here. Thus we will simply deduce the governing formulas from those obtained for the general case treated in this section. The student is advised not just to memorize the formulas but to observe how they are easily obtained from the formulas for the general case.

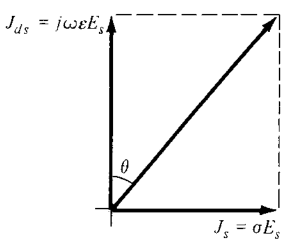

Figure 10.6 Loss angle of a lossy medium.

10.4 PLANE WAVES IN LOSSLESS DIELECTRICS

In a lossless dielectric, $\sigma \ll \omega\varepsilon$. It is a special case of that in Section 10.3 except that

$$\boxed{\sigma \simeq 0, \qquad \varepsilon = \varepsilon_o \varepsilon_r, \qquad \mu = \mu_o \mu_r} \tag{10.42}$$

Substituting these into eqs. (10.23) and (10.24) gives

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu\varepsilon} \tag{10.43a}$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}, \qquad \lambda = \frac{2\pi}{\beta} \tag{10.43b}$$

Also

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \angle 0^\circ \tag{10.44}$$

and thus **E** and **H** are in time phase with each other.

10.5 PLANE WAVES IN FREE SPACE

This is a special case of what we considered in Section 10.3. In this case,

$$\boxed{\sigma = 0, \qquad \varepsilon = \varepsilon_o, \qquad \mu = \mu_o} \tag{10.45}$$

This may also be regarded as a special case of Section 10.4. Thus we simply replace ε by ε_o and μ by μ_o in eq. (10.43) or we substitute eq. (10.45) directly into eqs. (10.23) and (10.24). Either way, we obtain

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu_o \varepsilon_o} = \frac{\omega}{c} \tag{10.46a}$$

$$u = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = c, \qquad \lambda = \frac{2\pi}{\beta} \tag{10.46b}$$

where $c \simeq 3 \times 10^8$ m/s, the speed of light in a vacuum. The fact that EM wave travels in free space at the speed of light is significant. It shows that light is the manifestation of an EM wave. In other words, light is characteristically electromagnetic.

By substituting the constitutive parameters in eq. (10.45) into eq. (10.33), $\theta_\eta = 0$ and $\eta = \eta_o$, where η_o is called the *intrinsic impedance of free space* and is given by

$$\boxed{\eta_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi \simeq 377\ \Omega} \tag{10.47}$$

$$\mathbf{E} = E_o \cos(\omega t - \beta z)\ \mathbf{a}_x \tag{10.48a}$$

then

$$\mathbf{H} = H_o \cos(\omega t - \beta z)\ \mathbf{a}_y = \frac{E_o}{\eta_o} \cos(\omega t - \beta z)\ \mathbf{a}_y \tag{10.48b}$$

The plots of **E** and **H** are shown in Figure 10.7(a). In general, if $\mathbf{a}_E$, $\mathbf{a}_H$, and $\mathbf{a}_k$ are unit vectors along the **E** field, the **H** field, and the direction of wave propagation; it can be shown that (see Problem 10.14)

$$\mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_H$$

or

$$\mathbf{a}_k \times \mathbf{a}_H = -\mathbf{a}_E$$

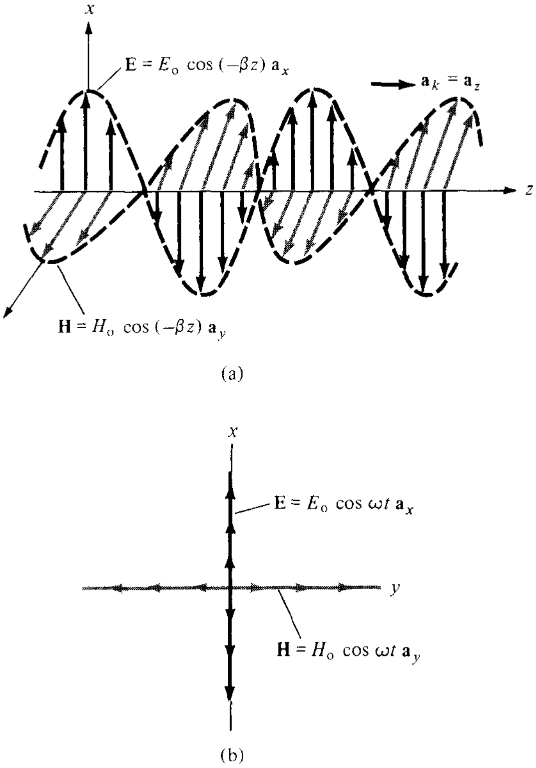

Figure 10.7 **(a)** Plot of **E** and **H** as functions of z at $t = 0$; **(b)** plot of **E** and **H** at $z = 0$. The arrows indicate instantaneous values.

or

$$\boxed{\mathbf{a}_E \times \mathbf{a}_H = \mathbf{a}_k} \tag{10.49}$$

Both **E** and **H** fields (or EM waves) are everywhere normal to the direction of wave propagation, $\mathbf{a}_k$. That means, the fields lie in a plane that is transverse or orthogonal to the direction of wave propagation. They form an EM wave that has no electric or magnetic field components along the direction of propagation; such a wave is called a *transverse electromagnetic* (TEM) wave. Each of **E** and **H** is called a *uniform plane wave* because **E** (or **H**) has the same magnitude throughout any transverse plane, defined by z = constant. The direction in which the electric field points is the *polarization* of a TEM wave.[3] The wave in eq. (10.29), for example, is polarized in the x-direction. This should be observed in Figure 10.7(b), where an illustration of uniform plane waves is given. A uniform plane wave cannot exist physically because it stretches to infinity and would represent an infinite energy. However, such waves are characteristically simple but fundamentally important. They serve as approximations to practical waves, such as from a radio antenna, at distances sufficiently far from radiating sources. Although our discussion after eq. (10.48) deals with free space, it also applies for any other isotropic medium.

10.6 PLANE WAVES IN GOOD CONDUCTORS

This is another special case of that considered in Section 10.3. A perfect, or good conductor, is one in which $\sigma \gg \omega\varepsilon$ so that $\sigma/\omega\varepsilon \to \infty$; that is,

$$\boxed{\sigma \simeq \infty, \qquad \varepsilon = \varepsilon_o, \qquad \mu = \mu_o\mu_r} \tag{10.50}$$

Hence, eqs. (10.23) and (10.24) become

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma} \tag{10.51a}$$

$$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}, \qquad \lambda = \frac{2\pi}{\beta} \tag{10.51b}$$

Also,

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}}\,\underline{/45^\circ} \tag{10.52}$$

and thus **E** leads **H** by 45°. If

$$\mathbf{E} = E_o e^{-\alpha z}\cos(\omega t - \beta z)\,\mathbf{a}_x \tag{10.53a}$$

[3] Some texts define polarization differently.

then

$$\mathbf{H} = \frac{E_o}{\sqrt{\frac{\omega\mu}{\sigma}}} e^{-\alpha z} \cos(\omega t - \beta z - 45°)\, \mathbf{a}_y \qquad (10.53b)$$

Therefore, as **E** (or **H**) wave travels in a conducting medium, its amplitude is attenuated by the factor $e^{-\alpha z}$. The distance δ, shown in Figure 10.8, through which the wave amplitude decreases by a factor e^{-1} (about 37%) is called *skin depth* or *penetration depth* of the medium; that is,

$$E_o e^{-\alpha\delta} = E_o e^{-1}$$

or

$$\boxed{\delta = \frac{1}{\alpha}} \qquad (10.54a)$$

The **skin depth** is a measure of the depth to which an **EM** wave can penetrate the medium.

Equation (10.54a) is generally valid for any material medium. For good conductors, eqs. (10.51a) and (10.54a) give

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \qquad (10.54b)$$

The illustration in Figure 10.8 for a good conductor is exaggerated. However, for a partially conducting medium, the skin depth can be considerably large. Note from eqs. (10.51a), (10.52), and (10.54b) that for a good conductor.

$$\eta = \frac{1}{\sigma\delta}\sqrt{2}\, e^{j\pi/4} = \frac{1+j}{\sigma\delta} \qquad (10.55)$$

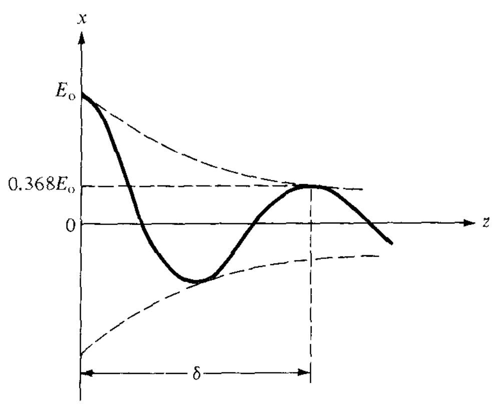

Figure 10.8 Illustration of skin depth.

TABLE 10.2 Skin Depth in Copper*

Frequency (Hz)	**10**	**60**	**100**	**500**	**10^4**	**10^8**	**10^{10}**
Skin depth (mm)	20.8	8.6	6.6	2.99	0.66	6.6×10^{-3}	6.6×10^{-4}

*For copper, $\sigma = 5.8 \times 10^7$ mhos/m, $\mu = \mu_o$, $\delta = 66.1/\sqrt{f}$ (in mm).

Also for good conductors, eq. (10.53a) can be written as

$$\mathbf{E} = E_o e^{-z/\delta} \cos\left(\omega t - \frac{z}{\delta}\right) \mathbf{a}_x$$

showing that δ measures the exponential damping of the wave as it travels through the conductor. The skin depth in copper at various frequencies is shown in Table 10.2. From the table, we notice that the skin depth decreases with increase in frequency. Thus, **E** and **H** can hardly propagate through good conductors.

The phenomenon whereby field intensity in a conductor rapidly decreases is known as *skin effect.* The fields and associated currents are confined to a very thin layer (the skin) of the conductor surface. For a wire of radius a, for example, it is a good approximation at high frequencies to assume that all of the current flows in the circular ring of thickness δ as shown in Figure 10.9. Skin effect appears in different guises in such problems as attenuation in waveguides, effective or ac resistance of transmission lines, and electromagnetic shielding. It is used to advantage in many applications. For example, because the skin depth in silver is very small, the difference in performance between a pure silver component and a silver-plated brass component is negligible, so silver plating is often used to reduce material cost of waveguide components. For the same reason, hollow tubular conductors are used instead of solid conductors in outdoor television antennas. Effective electromagnetic shielding of electrical devices can be provided by conductive enclosures a few skin depths in thickness.

The skin depth is useful in calculating the *ac resistance* due to skin effect. The resistance in eq. (5.16) is called the *dc resistance,* that is,

$$R_{dc} = \frac{\ell}{\sigma S} \tag{5.16}$$

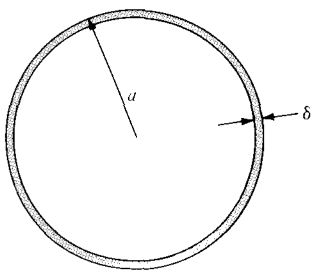

Figure 10.9 Skin depth at high frequencies, $\delta \ll a$.

We define the *surface or skin resistance* R_s (in Ω/m^2) as the real part of the η for a good conductor. Thus from eq. (10.55)

$$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\pi f \mu}{\sigma}} \tag{10.56}$$

This is the resistance of a unit width and unit length of the conductor. It is equivalent to the dc resistance for a unit length of the conductor having cross-sectional area $1 \times \delta$. Thus for a given width w and length ℓ, the ac resistance is calculated using the familiar dc resistance relation of eq. (5.16) and assuming a uniform current flow in the conductor of thickness δ, that is,

$$R_{ac} = \frac{\ell}{\sigma\delta w} = \frac{R_s \ell}{w} \tag{10.57}$$

where $S \simeq \delta w$. For a conductor wire of radius a (see Figure 10.9), $w = 2\pi a$, so

$$\frac{R_{ac}}{R_{dc}} = \frac{\dfrac{\ell}{\sigma 2\pi a\delta}}{\dfrac{\ell}{\sigma\pi a^2}} = \frac{a}{2\delta}$$

Since $\delta \ll a$ at high frequencies, this shows that R_{ac} is far greater than R_{dc}. In general, the ratio of the ac to the dc resistance starts at 1.0 for dc and very low frequencies and increases as the frequency increases. Also, although the bulk of the current is nonuniformly distributed over a thickness of 5δ of the conductor, the power loss is the same as though it were uniformly distributed over a thickness of δ and zero elsewhere. This is one more reason why δ is referred to as the skin depth.

EXAMPLE 10.2

A lossy dielectric has an intrinsic impedance of $200 \angle 30^\circ\ \Omega$ at a particular frequency. If, at that frequency, the plane wave propagating through the dielectric has the magnetic field component

$$\mathbf{H} = 10\, e^{-\alpha x} \cos\left(\omega t - \frac{1}{2}x\right) \mathbf{a}_y \text{ A/m}$$

find $\mathbf{E}$ and α. Determine the skin depth and wave polarization.

Solution:

The given wave travels along $\mathbf{a}_x$ so that $\mathbf{a}_k = \mathbf{a}_x$; $\mathbf{a}_H = \mathbf{a}_y$, so

$$-\mathbf{a}_E = \mathbf{a}_k \times \mathbf{a}_H = \mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$$

or

$$\mathbf{a}_E = -\mathbf{a}_z$$

Also $H_o = 10$, so

$$\frac{E_o}{H_o} = \eta = 200 \underline{/30^\circ} = 200\, e^{j\pi/6} \rightarrow E_o = 2000 e^{j\pi/6}$$

Except for the amplitude and phase difference, **E** and **H** always have the same form. Hence

$$\mathbf{E} = \text{Re}\,(2000 e^{j\pi/6} e^{-\gamma x} e^{j\omega t} \mathbf{a}_E)$$

or

$$\mathbf{E} = -2e^{-\alpha x} \cos\left(\omega t - \frac{x}{2} + \frac{\pi}{6}\right) \mathbf{a}_z \text{ kV/m}$$

Knowing that $\beta = 1/2$, we need to determine α. Since

$$\alpha = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]}$$

and

$$\beta = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]}$$

$$\frac{\alpha}{\beta} = \left[\frac{\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1}{\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1}\right]^{1/2}$$

But $\dfrac{\sigma}{\omega\varepsilon} = \tan 2\theta_\eta = \tan 60^\circ = \sqrt{3}$. Hence,

$$\frac{\alpha}{\beta} = \left[\frac{2-1}{2+1}\right]^{1/2} = \frac{1}{\sqrt{3}}$$

or

$$\alpha = \frac{\beta}{\sqrt{3}} = \frac{1}{2\sqrt{3}} = 0.2887 \text{ Np/m}$$

and

$$\delta = \frac{1}{\alpha} = 2\sqrt{3} = 3.464 \text{ m}$$

The wave has an E_z component; hence it is polarized along the z-direction.

PRACTICE EXERCISE 10.2

A plane wave propagating through a medium with $\varepsilon_r = 8$, $\mu_r = 2$ has $\mathbf{E} = 0.5\, e^{-z/3} \sin(10^8 t - \beta z)\, \mathbf{a}_x$ V/m. Determine

(a) β

(b) The loss tangent

(c) Wave impedance

(d) Wave velocity

(e) **H** field

Answer: (a) 1.374 rad/m, (b) 0.5154, (c) 177.72 $\angle 13.63°$ Ω, (d) 7.278×10^7 m/s, (e) $2.817 e^{-z/3} \sin(10^8 t - \beta z - 13.63°)\mathbf{a}_y$ mA/m.

EXAMPLE 10.3

In a lossless medium for which $\eta = 60\pi$, $\mu_r = 1$, and $\mathbf{H} = -0.1 \cos(\omega t - z)\, \mathbf{a}_x + 0.5 \sin(\omega t - z)\mathbf{a}_y$ A/m, calculate ε_r, ω, and $\mathbf{E}$.

Solution:

In this case, $\sigma = 0$, $\alpha = 0$, and $\beta = 1$, so

$$\eta = \sqrt{\mu/\varepsilon} = \sqrt{\frac{\mu_o}{\varepsilon_o}} \sqrt{\frac{\mu_r}{\varepsilon_r}} = \frac{120\pi}{\sqrt{\varepsilon_r}}$$

or

$$\sqrt{\varepsilon_r} = \frac{120\pi}{\eta} = \frac{120\pi}{60\pi} = 2 \quad \rightarrow \quad \varepsilon_r = 4$$

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\varepsilon_o}\,\sqrt{\mu_r\varepsilon_r} = \frac{\omega}{c}\sqrt{4} = \frac{2\omega}{c}$$

or

$$\omega = \frac{\beta c}{2} = \frac{1\,(3 \times 10^8)}{2} = 1.5 \times 10^8 \text{ rad/s}$$

From the given **H** field, **E** can be calculated in two ways: using the techniques (based on Maxwell's equations) developed in this chapter or directly using Maxwell's equations as in the last chapter.

Method 1: To use the techniques developed in this chapter, we let

$$\mathbf{E} = \mathbf{H}_1 + \mathbf{H}_2$$

where $\mathbf{H}_1 = -0.1 \cos(\omega t - z)\, \mathbf{a}_x$ and $\mathbf{H}_2 = 0.5 \sin(\omega t - z)\, \mathbf{a}_y$ and the corresponding electric field

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

where $\mathbf{E}_1 = E_{1o} \cos(\omega t - z)\, \mathbf{a}_{E_1}$ and $\mathbf{E}_2 = E_{2o} \sin(\omega t - z)\, \mathbf{a}_{E_2}$. Notice that although $\mathbf{H}$ has components along $\mathbf{a}_x$ and $\mathbf{a}_y$, it has no component along the direction of propagation; it is therefore a TEM wave.

For $\mathbf{E}_1$,

$$\mathbf{a}_{E_1} = -(\mathbf{a}_k \times \mathbf{a}_{H_1}) = -(\mathbf{a}_z \times -\mathbf{a}_x) = \mathbf{a}_y$$

$$E_{1o} = \eta H_{1o} = 60\pi\,(0.1) = 6\pi$$

Hence

$$\mathbf{E}_1 = 6\pi \cos(\omega t - z)\, \mathbf{a}_y$$

For $\mathbf{E}_2$,

$$\mathbf{a}_{E_2} = -(\mathbf{a}_k \times \mathbf{a}_{H_2}) = -(\mathbf{a}_z \times \mathbf{a}_y) = \mathbf{a}_x$$

$$E_{2o} = \eta H_{2o} = 60\pi\,(0.5) = 30\pi$$

Hence

$$\mathbf{E}_2 = 30\pi \sin(\omega t - z)\mathbf{a}_x$$

Adding $\mathbf{E}_1$ and $\mathbf{E}_2$ gives $\mathbf{E}$; that is,

$$\mathbf{E} = 94.25 \sin(1.5 \times 10^8 t - z)\, \mathbf{a}_x + 18.85 \cos(1.5 \times 10^8 t - z)\, \mathbf{a}_y \text{ V/m}$$

Method 2: We may apply Maxwell's equations directly.

$$\nabla \times \mathbf{H} = \underbrace{\sigma \mathbf{E}}_{0} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad \rightarrow \quad \mathbf{E} = \frac{1}{\varepsilon} \int \nabla \times H\, dt$$

because $\sigma = 0$. But

$$\nabla \times \mathbf{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x(z) & H_y(z) & 0 \end{vmatrix} = -\frac{\partial H_y}{\partial z} \mathbf{a}_x + \frac{\partial H_x}{\partial z} \mathbf{a}_y$$

$$= H_{2o} \cos(\omega t - z)\, \mathbf{a}_x + H_{1o} \sin(\omega t - z)\mathbf{a}_y$$

where $H_{1o} = -0.1$ and $H_{2o} = 0.5$. Hence

$$\mathbf{E} = \frac{1}{\varepsilon} \int \nabla \times \mathbf{H}\, dt = \frac{H_{2o}}{\varepsilon\omega} \sin(\omega t - z)\, \mathbf{a}_x - \frac{H_{1o}}{\varepsilon\omega} \cos(\omega t - z)\, \mathbf{a}_y$$

$$= 94.25 \sin(\omega t - z)\, \mathbf{a}_x + 18.85 \cos(\omega t - z)\, \mathbf{a}_y \text{ V/m}$$

as expected.

PRACTICE EXERCISE 10.3

A plane wave in a nonmagnetic medium has $\mathbf{E} = 50 \sin(10^8 t + 2z)\,\mathbf{a}_y$ V/m. Find

(a) The direction of wave propagation

(b) λ, f, and ε_r

(c) $\mathbf{H}$

Answer: (a) along $-z$ direction, (b) 3.142 m, 15.92 MHz, 36, (c) 0.7958 $\sin(10^8 t + 2z)\,\mathbf{a}_x$ A/m.

EXAMPLE 10.4

A uniform plane wave propagating in a medium has

$$\mathbf{E} = 2e^{-\alpha z} \sin(10^8 t - \beta z)\,\mathbf{a}_y \text{ V/m.}$$

If the medium is characterized by $\varepsilon_r = 1$, $\mu_r = 20$, and $\sigma = 3$ mhos/m, find α, β, and $\mathbf{H}$.

Solution:

We need to determine the loss tangent to be able to tell whether the medium is a lossy dielectric or a good conductor.

$$\frac{\sigma}{\omega\varepsilon} = \frac{3}{10^8 \times 1 \times \dfrac{10^{-9}}{36\pi}} = 3393 \gg 1$$

showing that the medium may be regarded as a good conductor at the frequency of operation. Hence,

$$\alpha = \beta = \sqrt{\frac{\mu\omega\sigma}{2}} = \left[\frac{4\pi \times 10^{-7} \times 20(10^8)(3)}{2}\right]^{1/2}$$
$$= 61.4$$
$$\alpha = 61.4 \text{ Np/m}, \qquad \beta = 61.4 \text{ rad/m}$$

Also

$$|\eta| = \sqrt{\frac{\mu\omega}{\sigma}} = \left[\frac{4\pi \times 10^{-7} \times 20(10^8)}{3}\right]^{1/2}$$
$$= \sqrt{\frac{800\pi}{3}}$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = 3393 \quad \rightarrow \quad \theta_\eta = 45^\circ = \pi/4$$

Hence

$$\mathbf{H} = H_o e^{-\alpha z} \sin\left(\omega t - \beta z - \frac{\pi}{4}\right) \mathbf{a}_H$$

where

$$\mathbf{a}_H = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_y = -\mathbf{a}_x$$

and

$$H_o = \frac{E_o}{|\eta|} = 2\sqrt{\frac{3}{800\pi}} = 69.1 \times 10^{-3}$$

Thus

$$\mathbf{H} = -69.1\, e^{-61.4z} \sin\left(10^8 t - 61.42z - \frac{\pi}{4}\right) \mathbf{a}_x \text{ mA/m}$$

PRACTICE EXERCISE 10.4

A plane wave traveling in the $+y$-direction in a lossy medium ($\varepsilon_r = 4$, $\mu_r = 1$, $\sigma = 10^{-2}$ mhos/m) has $\mathbf{E} = 30 \cos(10^9 \pi\, t + \pi/4)\, \mathbf{a}_z$ V/m at $y = 0$. Find

(a) $\mathbf{E}$ at $y = 1$ m, $t = 2$ ns

(b) The distance traveled by the wave to have a phase shift of 10°

(c) The distance traveled by the wave to have its amplitude reduced by 40%

(d) $\mathbf{H}$ at $y = 2$ m, $t = 2$ ns

Answer: (a) $2.787\mathbf{a}_z$ V/m, (b) 8.325 mm, (c) 542 mm, (d) $-4.71\mathbf{a}_x$ mA/m

XAMPLE 10.5

A plane wave $\mathbf{E} = E_o \cos(\omega t - \beta z)\, \mathbf{a}_x$ is incident on a good conductor at $z = 0$. Find the current density in the conductor.

Solution:

Since the current density $\mathbf{J} = \sigma \mathbf{E}$, we expect $\mathbf{J}$ to satisfy the wave equation in eq. (10.17), that is,

$$\nabla^2 \mathbf{J}_s - \gamma^2 \mathbf{J}_s = 0$$

Also the incident $\mathbf{E}$ has only an x-component and varies with z. Hence $\mathbf{J} = J_x(z, t)\, \mathbf{a}_x$ and

$$\frac{d^2}{dz^2} J_{sx} - \gamma^2 J_{sx} = 0$$

which is an ordinary differential equation with solution (see Case 2 of Example 6.5)

$$J_{sx} = Ae^{-\gamma z} + Be^{+\gamma z}$$

The constant B must be zero because J_{sx} is finite as $z \rightarrow \infty$. But in a good conductor, $\sigma \gg \omega\varepsilon$ so that $\alpha = \beta = 1/\delta$. Hence

$$\gamma = \alpha + j\beta = \alpha(1 + j) = \frac{(1 + j)}{\delta}$$

and

$$J_{sx} = Ae^{-z(1+j)/\delta}$$

or

$$J_{sx} = J_{sx}(0)\, e^{-z(1+j)/\delta}$$

where J_{sx} (0) is the current density on the conductor surface.

PRACTICE EXERCISE 10.5

Due to the current density of Example 10.5, find the magnitude of the total current through a strip of the conductor of infinite depth along z and width w along y.

Answer: $\dfrac{J_{sx}(0)w\delta}{\sqrt{2}}$

EXAMPLE 10.6

For the copper coaxial cable of Figure 7.12, let $a = 2$ mm, $b = 6$ mm, and $t = 1$ mm. Calculate the resistance of 2 m length of the cable at dc and at 100 MHz.

Solution:

Let

$$R = R_o + R_i$$

where R_o and R_i are the resistances of the inner and outer conductors.

At dc,

$$R_i = \frac{\ell}{\sigma S} = \frac{\ell}{\sigma \pi a^2} = \frac{2}{5.8 \times 10^7 \pi [2 \times 10^{-3}]^2} = 2.744 \text{ m}\Omega$$

$$\begin{aligned} R_o &= \frac{\ell}{\sigma S} = \frac{\ell}{\sigma\pi[[b + t]^2 - b^2]} = \frac{\ell}{\sigma\pi[t^2 + 2bt]} \\ &= \frac{2}{5.8 \times 10^7 \pi\, [1 + 12] \times 10^{-6}} \\ &= 0.8429 \text{ m}\Omega \end{aligned}$$

Hence $R_{dc} = 2.744 + 0.8429 = 3.587 \text{ m}\Omega$

At $f = 100$ MHz,

$$R_i = \frac{R_s \ell}{w} = \frac{\ell}{\sigma\delta 2\pi a} = \frac{\ell}{2\pi a}\sqrt{\frac{\pi f \mu}{\sigma}}$$

$$= \frac{2}{2\pi \times 2 \times 10^{-3}}\sqrt{\frac{\pi \times 10^8 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}}$$

$$= 0.41\ \Omega$$

Since $\delta = 6.6\ \mu\text{m} \ll t = 1$ mm, $w = 2\pi b$ for the outer conductor. Hence,

$$R_o = \frac{R_s \ell}{w} = \frac{\ell}{2\pi b}\sqrt{\frac{\pi f \mu}{\sigma}}$$

$$= \frac{2}{2\pi \times 6 \times 10^{-3}}\sqrt{\frac{\pi \times 10^8 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}}$$

$$= 0.1384\ \Omega$$

Hence,

$$R_{ac} = 0.41 + 0.1384 = 0.5484\ \Omega$$

which is about 150 times greater than R_{dc}. Thus, for the same effective current i, the ohmic loss (i^2R) of the cable at 100 MHz is far greater than the dc power loss by a factor of 150.

PRACTICE EXERCISE 10.6

For an aluminum wire having a diameter 2.6 mm, calculate the ratio of ac to dc resistance at

(a) 10 MHz

(b) 2 GHz

Answer: (a) 24.16, (b) 341.7.

0.7 POWER AND THE POYNTING VECTOR

As mentioned before, energy can be transported from one point (where a transmitter is located) to another point (with a receiver) by means of EM waves. The rate of such energy transportation can be obtained from Maxwell's equations:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{10.58a}$$

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \tag{10.58b}$$

Dotting both sides of eq. (10.58b) with **E** gives

$$\mathbf{E} \cdot (\nabla \times \mathbf{H}) = \sigma E^2 + \mathbf{E} \cdot \varepsilon \frac{\partial \mathbf{E}}{\varepsilon t} \tag{10.59}$$

But for any vector fields **A** and **B** (see Appendix A.10)

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}).$$

Applying this vector identity to eq. (10.59) (letting $\mathbf{A} = \mathbf{H}$ and $\mathbf{B} = \mathbf{E}$) gives

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) + \nabla \cdot (\mathbf{H} \times \mathbf{E}) = \sigma E^2 + \mathbf{E} \cdot \varepsilon \frac{\partial \mathbf{E}}{\partial t} \tag{10.60}$$

From eq. (10.58a),

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) = \mathbf{H} \cdot \left(-\mu \frac{\partial \mathbf{H}}{\partial t} \right) = -\frac{\mu}{2} \frac{\partial}{\partial t} (\mathbf{H} \cdot \mathbf{H}) \tag{10.61}$$

and thus eq. (10.60) becomes

$$-\frac{\mu}{2} \frac{\partial H^2}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \sigma E^2 + \frac{1}{2} \varepsilon \frac{\partial E^2}{\partial t}$$

Rearranging terms and taking the volume integral of both sides,

$$\int_v \nabla \cdot (\mathbf{E} \times \mathbf{H}) \, dv = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right] dv - \int_v \sigma E^2 \, dv \tag{10.62}$$

Applying the divergence theorem to the left-hand side gives

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right] dv - \int_v \sigma E^2 \, dv \tag{10.63}$$

↓		↓		↓	
Total power leaving the volume	=	Rate of decrease in energy stored in electric and magnetic fields	−	Ohmic power dissipated	(10.64)

Equation (10.63) is referred to as *Poynting's theorem.*[4] The various terms in the equation are identified using energy-conservation arguments for EM fields. The first term on the right-hand side of eq. (10.63) is interpreted as the rate of decrease in energy stored in the electric and magnetic fields. The second term is the power dissipated due to the fact that the medium is conducting ($\sigma \neq 0$). The quantity $\mathbf{E} \times \mathbf{H}$ on the left-hand side of eq. (10.63) is known as the *Poynting vector* $\mathcal{P}$ in watts per square meter (W/m^2); that is,

$$\boxed{\mathcal{P} = \mathbf{E} \times \mathbf{H}} \tag{10.65}$$

[4]After J. H. Poynting, "On the transfer of energy in the electromagnetic field," *Phil. Trans.,* vol. 174, 1883, p. 343.

It represents the instantaneous power density vector associated with the EM field at a given point. The integration of the Poynting vector over any closed surface gives the net power flowing out of that surface.

Poynting's theorem states that the net power flowing out of a given volume v is equal to the time rate of decrease in the energy stored within v minus the conduction losses.

The theorem is illustrated in Figure 10.10.

It should be noted that $\mathcal{P}$ is normal to both **E** and **H** and is therefore along the direction of wave propagation $\mathbf{a}_k$ for uniform plane waves. Thus

$$\mathbf{a}_k = \mathbf{a}_E \times \mathbf{a}_H \tag{10.49}$$

The fact that $\mathcal{P}$ points along $\mathbf{a}_k$ causes $\mathcal{P}$ to be regarded derisively as a "pointing" vector.

Again, if we assume that

$$\mathbf{E}(z, t) = E_o e^{-\alpha z} \cos(\omega t - \beta z)\, \mathbf{a}_x$$

then

$$\mathbf{H}(z, t) = \frac{E_o}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\, \mathbf{a}_y$$

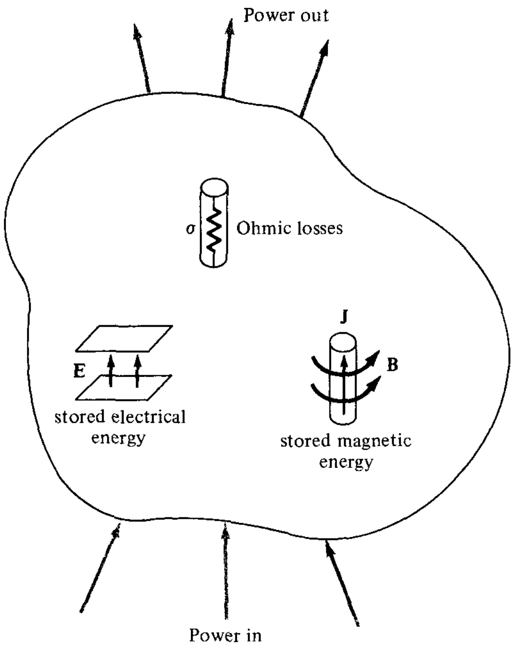

Figure 10.10 Illustration of power balance for EM fields.

and

$$\begin{aligned}\mathcal{P}(z, t) &= \frac{E_o^2}{|\eta|} e^{-2\alpha z} \cos(\omega t - \beta z) \cos(\omega t - \beta z - \theta_\eta)\, \mathbf{a}_z \\ &= \frac{E_o^2}{2|\eta|} e^{-2\alpha z} [\cos \theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)]\, \mathbf{a}_z\end{aligned} \tag{10.66}$$

since $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$. To determine the time-average Poynting vector $\mathcal{P}_{ave}(z)$ (in W/m^2), which is of more practical value than the instantaneous Poynting vector $\mathcal{P}(z, t)$, we integrate eq. (10.66) over the period $T = 2\pi/\omega$; that is,

$$\mathcal{P}_{ave}(z) = \frac{1}{T} \int_0^T \mathcal{P}(z, t)\, dt \tag{10.67}$$

It can be shown (see Prob. 10.28) that this is equivalent to

$$\boxed{\mathcal{P}_{ave}(z) = \frac{1}{2} \text{Re}\,(\mathbf{E}_s \times \mathbf{H}_s^*)} \tag{10.68}$$

By substituting eq. (10.66) into eq. (10.67), we obtain

$$\boxed{\mathcal{P}_{ave}(z) = \frac{E_o^2}{2|\eta|} e^{-2\alpha z} \cos \theta_\eta\, \mathbf{a}_z} \tag{10.69}$$

The total time-average power crossing a given surface S is given by

$$\boxed{P_{ave} = \int_S \mathcal{P}_{ave} \cdot d\mathbf{S}} \tag{10.70}$$

We should note the difference between $\mathcal{P}$, $\mathcal{P}_{ave}$, and P_{ave}. $\mathcal{P}(x, y, z, t)$ is the Poynting vector in watts/meter and is time varying. $\mathcal{P}_{ave}(x, y, z)$ also in watts/meter is the time average of the Poynting vector $\mathcal{P}$; it is a vector but is time invariant. P_{ave} is a total time-average power through a surface in watts; it is a scalar.

EXAMPLE 10.7

In a nonmagnetic medium

$$\mathbf{E} = 4 \sin(2\pi \times 10^7 t - 0.8x)\, \mathbf{a}_z \text{ V/m}$$

Find

(a) ε_r, η
(b) The time-average power carried by the wave
(c) The total power crossing 100 cm^2 of plane $2x + y = 5$

Solution:

(a) Since $\alpha = 0$ and $\beta \neq \omega/c$, the medium is not free space but a lossless medium.

$$\beta = 0.8, \quad \omega = 2\pi \times 10^7, \quad \mu = \mu_o \text{ (nonmagnetic)}, \quad \varepsilon = \varepsilon_o\varepsilon_r$$

Hence

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\varepsilon_o\varepsilon_r} = \frac{\omega}{c}\sqrt{\varepsilon_r}$$

or

$$\sqrt{\varepsilon_r} = \frac{\beta c}{\omega} = \frac{0.8(3 \times 10^8)}{2\pi \times 10^7} = \frac{12}{\pi}$$

$$\varepsilon_r = 14.59$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_o}{\varepsilon_o\varepsilon_r}} = \frac{120\pi}{\sqrt{\varepsilon_r}} = 120\pi \cdot \frac{\pi}{12} = 10\pi^2$$
$$= 98.7\ \Omega$$

(b) $\mathcal{P} = \mathbf{E} \times \mathbf{H} = \dfrac{E_o^2}{\eta}\sin^2(\omega t - \beta x)\,\mathbf{a}_x$

$$\mathcal{P}_{\text{ave}} = \frac{1}{T}\int_0^T \mathcal{P}\,dt = \frac{E_o^2}{2\eta}\mathbf{a}_x = \frac{16}{2 \times 10\pi^2}\mathbf{a}_x$$
$$= 81\,\mathbf{a}_x \text{ mW/m}^2$$

(c) On plane $2x + y = 5$ (see Example 3.5 or 8.5),

$$\mathbf{a}_n = \frac{2\mathbf{a}_x + \mathbf{a}_y}{\sqrt{5}}$$

Hence the total power is

$$P_{\text{ave}} = \int \mathcal{P}_{\text{ave}} \cdot d\mathbf{S} = \mathcal{P}_{\text{ave}} \cdot S\,\mathbf{a}_n$$
$$= (81 \times 10^{-3}\mathbf{a}_x) \cdot (100 \times 10^{-4})\left[\frac{2\mathbf{a}_x + \mathbf{a}_y}{\sqrt{5}}\right]$$
$$= \frac{162 \times 10^{-5}}{\sqrt{5}} = 724.5\ \mu\text{W}$$

PRACTICE EXERCISE 10.7

In free space, $\mathbf{H} = 0.2 \cos(\omega t - \beta x)\, \mathbf{a}_z$ A/m. Find the total power passing through:

(a) A square plate of side 10 cm on plane $x + z = 1$

(b) A circular disc of radius 5 cm on plane $x = 1$.

Answer: (a) 0, (b) 59.22 mW.

10.8 REFLECTION OF A PLANE WAVE AT NORMAL INCIDENCE

So far, we have considered uniform plane waves traveling in unbounded, homogeneous media. When a plane wave from one medium meets a different medium, it is partly reflected and partly transmitted. The proportion of the incident wave that is reflected or transmitted depends on the constitutive parameters (ε, μ, σ) of the two media involved. Here we will assume that the incident wave plane is normal to the boundary between the media; oblique incidence of plane waves will be covered in the next section after we understand the simpler case of normal incidence.

Suppose that a plane wave propagating along the $+z$-direction is incident normally on the boundary $z = 0$ between medium 1 ($z < 0$) characterized by σ_1, ε_1, μ_1 and medium 2 ($z > 0$) characterized by σ_2, ε_2, μ_2, as shown in Figure 10.11. In the figure, subscripts i, r, and t denote incident, reflected, and transmitted waves, respectively. The incident, reflected, and transmitted waves shown in Figure 10.11 are obtained as follows:

Incident Wave:

($\mathbf{E}_i$, $\mathbf{H}_i$) is traveling along $+\mathbf{a}_z$ in medium 1. If we suppress the time factor $e^{j\omega t}$ and assume that

$$\mathbf{E}_{is}(z) = E_{io} e^{-\gamma_1 z}\, \mathbf{a}_x \tag{10.71}$$

then

$$\mathbf{H}_{is}(z) = H_{io} e^{-\gamma_1 z}\, \mathbf{a}_y = \frac{E_{io}}{\eta_1} e^{-\gamma_1 z}\, \mathbf{a}_y \tag{10.72}$$

Reflected Wave:

($\mathbf{E}_r$, $\mathbf{H}_r$) is traveling along $-\mathbf{a}_z$ in medium 1. If

$$\mathbf{E}_{rs}(z) = E_{ro} e^{\gamma_1 z}\, \mathbf{a}_x \tag{10.73}$$

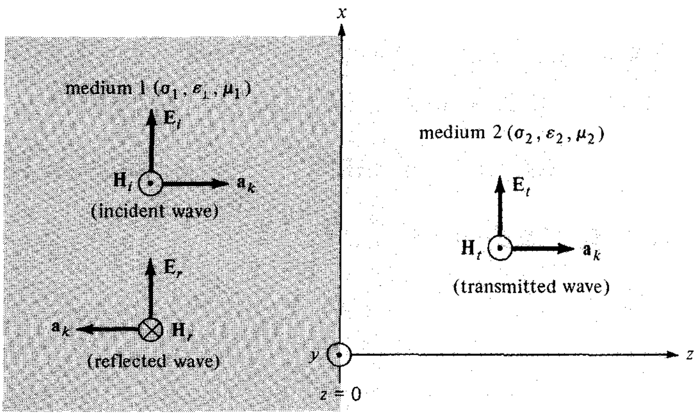

Figure 10.11 A plane wave incident normally on an interface between two different media.

then

$$\mathbf{H}_{rs}(z) = H_{ro}\, e^{\gamma_1 z}(-\mathbf{a}_y) = -\frac{E_{ro}}{\eta_1} e^{\gamma_1 z}\, \mathbf{a}_y \tag{10.74}$$

where $\mathbf{E}_{rs}$ has been assumed to be along $\mathbf{a}_x$; we will consistently assume that for normal incident $\mathbf{E}_i$, $\mathbf{E}_r$, and $\mathbf{E}_t$ have the same polarization.

Transmitted Wave:

$(\mathbf{E}_t, \mathbf{H}_t)$ is traveling along $+\mathbf{a}_z$ in medium 2. If

$$\mathbf{E}_{ts}(z) = E_{to}\, e^{-\gamma_2 z}\, \mathbf{a}_x \tag{10.75}$$

then

$$\mathbf{H}_{ts}(z) = H_{to}\, e^{-\gamma_2 z}\, \mathbf{a}_y = \frac{E_{to}}{\eta_2} e^{-\gamma_2 z}\, \mathbf{a}_y \tag{10.76}$$

In eqs. (10.71) to (10.76), E_{io}, E_{ro}, and E_{to} are, respectively, the magnitudes of the incident, reflected, and transmitted electric fields at $z = 0$.

Notice from Figure 10.11 that the total field in medium 1 comprises both the incident and reflected fields, whereas medium 2 has only the transmitted field, that is,

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r, \qquad \mathbf{H}_1 = \mathbf{H}_i + \mathbf{H}_r$$

$$\mathbf{E}_2 = \mathbf{E}_t, \qquad \mathbf{H}_2 = \mathbf{H}_t$$

At the interface $z = 0$, the boundary conditions require that the tangential components of $\mathbf{E}$ and $\mathbf{H}$ fields must be continuous. Since the waves are transverse, $\mathbf{E}$ and $\mathbf{H}$ fields are entirely tangential to the interface. Hence at $z = 0$, $\mathbf{E}_{1\tan} = \mathbf{E}_{2\tan}$ and $\mathbf{H}_{1\tan} = \mathbf{H}_{2\tan}$ imply that

$$\mathbf{E}_i(0) + \mathbf{E}_r(0) = \mathbf{E}_t(0) \quad \rightarrow \quad E_{io} + E_{ro} = E_{to} \tag{10.77}$$

$$\mathbf{H}_i(0) + \mathbf{H}_r(0) = \mathbf{H}_t(0) \quad \rightarrow \quad \frac{1}{\eta_1}(E_{io} - \mathbf{E}_{ro}) = \frac{E_{to}}{\eta_2} \tag{10.78}$$

From eqs. (10.77) and (10.78), we obtain

$$E_{ro} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{io} \tag{10.79}$$

and

$$E_{to} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{io} \tag{10.80}$$

We now define the *reflection coefficient* Γ and the *transmission coefficient* τ from eqs. (10.79) and (10.80) as

$$\boxed{\Gamma = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}} \tag{10.81a}$$

or

$$E_{ro} = \Gamma E_{io} \tag{10.81b}$$

and

$$\boxed{\tau = \frac{E_{to}}{E_{io}} = \frac{2\eta_2}{\eta_2 + \eta_1}} \tag{10.82a}$$

or

$$E_{to} = \tau E_{io} \tag{10.82b}$$

Note that

1. $1 + \Gamma = \tau$
2. Both Γ and τ are dimensionless and may be complex.
3. $0 \leq |\Gamma| \leq 1$ (10.83)

The case considered above is the general case. Let us now consider a special case when medium 1 is a perfect dielectric (lossless, $\sigma_1 = 0$) and medium 2 is a perfect conductor ($\sigma_2 \simeq \infty$). For this case, $\eta_2 = 0$; hence, $\Gamma = -1$, and $\tau = 0$, showing that the wave is totally reflected. This should be expected because fields in a perfect conductor must vanish, so there can be no transmitted wave ($\mathbf{E}_2 = 0$). The totally reflected wave combines with the incident wave to form a *standing wave*. A standing wave "stands" and does not travel; it consists of two traveling waves ($\mathbf{E}_i$ and $\mathbf{E}_r$) of equal amplitudes but in opposite directions. Combining eqs. (10.71) and (10.73) gives the standing wave in medium 1 as

$$\mathbf{E}_{1s} = \mathbf{E}_{is} + \mathbf{E}_{rs} = (E_{io}e^{-\gamma_1 z} + E_{ro}e^{\gamma_1 z})\,\mathbf{a}_x \tag{10.84}$$

But

$$\Gamma = \frac{E_{ro}}{E_{io}} = -1, \sigma_1 = 0, \alpha_1 = 0, \gamma_1 = j\beta_1$$

Hence,

$$\mathbf{E}_{1s} = -E_{io}(e^{j\beta_1 z} - e^{-j\beta_1 z})\,\mathbf{a}_x$$

or

$$\mathbf{E}_{1s} = -2jE_{io}\sin\beta_1 z\,\mathbf{a}_x \tag{10.85}$$

Thus

$$\mathbf{E}_1 = \text{Re}\,(\mathbf{E}_{1s}e^{j\omega t})$$

or

$$\boxed{\mathbf{E}_1 = 2E_{io}\sin\beta_1 z\sin\omega t\,\mathbf{a}_x} \tag{10.86}$$

By taking similar steps, it can be shown that the magnetic field component of the wave is

$$\boxed{\mathbf{H}_1 = \frac{2E_{io}}{\eta_1}\cos\beta_1 z\cos\omega t\,\mathbf{a}_y} \tag{10.87}$$

A sketch of the standing wave in eq. (10.86) is presented in Figure 10.12 for $t = 0, T/8, T/4, 3T/8, T/2$, and so on, where $T = 2\pi/\omega$. From the figure, we notice that the wave does not travel but oscillates.

When media 1 and 2 are both lossless we have another special case ($\sigma_1 = 0 = \sigma_2$). In this case, η_1 and η_2 are real and so are Γ and τ. Let us consider the following cases:

CASE A.

If $\eta_2 > \eta_1, \Gamma > 0$. Again there is a standing wave in medium 1 but there is also a transmitted wave in medium 2. However, the incident and reflected waves have amplitudes that are not equal in magnitude. It can be shown that the maximum values of $|\mathbf{E}_1|$ occur at

$$-\beta_1 z_{\max} = n\pi$$

or

$$z_{\max} = -\frac{n\pi}{\beta_1} = -\frac{n\lambda_1}{2}, \qquad n = 0, 1, 2, \ldots \tag{10.88}$$

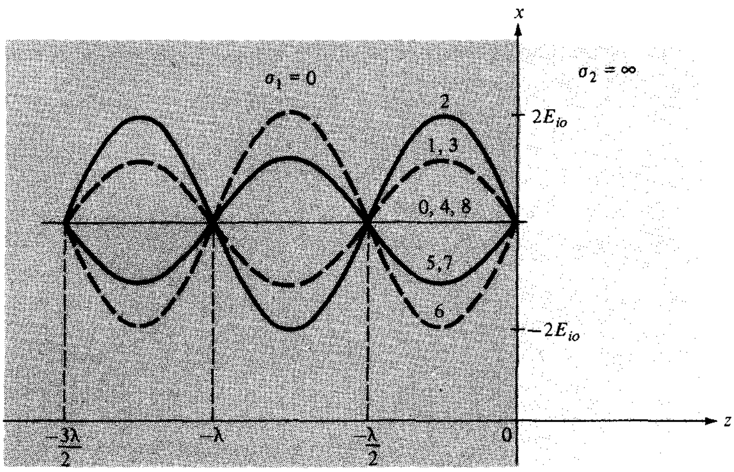

Figure 10.12 Standing waves $E = 2E_{io} \sin \beta_1 z \sin \omega t\, \mathbf{a}_x$; curves 0, 1, 2, 3, 4, . . . are, respectively, at times $t = 0, T/8, T/4, 3T/8, T/2, \ldots$; $\lambda = 2\pi/\beta_1$.

and the minimum values of $|\mathbf{E}_1|$ occur at

$$-\beta_1 z_{\min} = (2n + 1)\frac{\pi}{2}$$

or

$$z_{\min} = -\frac{(2n + 1)\pi}{2\beta_1} = -\frac{(2n + 1)}{4}\lambda_1, \qquad n = 0, 1, 2, \ldots \tag{10.89}$$

CASE B.

If $\eta_2 < \eta_1$, $\Gamma < 0$. For this case, the locations of $|\mathbf{E}_1|$ maximum are given by eq. (10.89) whereas those of $|\mathbf{E}_1|$ minimum are given by eq. (10.88). All these are illustrated in Figure 10.13. Note that

1. $|\mathbf{H}_1|$ minimum occurs whenever there is $|\mathbf{E}_1|$ maximum and vice versa.
2. The transmitted wave (not shown in Figure 10.13) in medium 2 is a purely traveling wave and consequently there are no maxima or minima in this region.

The ratio of $|\mathbf{E}_1|_{\max}$ to $|\mathbf{E}_1|_{\min}$ (or $|\mathbf{H}_1|_{\max}$ to $|\mathbf{H}_1|_{\min}$) is called the *standing-wave ratio* *s;* that is,

$$s = \frac{|\mathbf{E}_1|_{\max}}{|\mathbf{E}_1|_{\min}} = \frac{|\mathbf{H}_1|_{\max}}{|\mathbf{H}_1|_{\min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \tag{10.90}$$

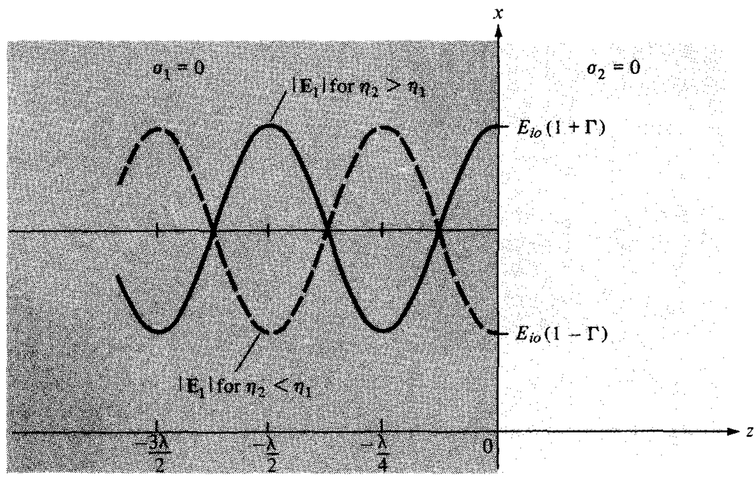

Figure 10.13 Standing waves due to reflection at an interface between two lossless media; $\lambda = 2\pi/\beta_1$.

or

$$|\Gamma| = \frac{s - 1}{s + 1} \tag{10.91}$$

Since $|\Gamma| \leq 1$, it follows that $1 \leq s \leq \infty$. The standing-wave ratio is dimensionless and it is customarily expressed in decibels (dB) as

$$s \text{ in dB} = 20 \log_{10} s \tag{10.92}$$

EXAMPLE 10.8

In free space ($z \leq 0$), a plane wave with

$$\mathbf{H} = 10 \cos (10^8 t - \beta z) \, \mathbf{a}_x \text{ mA/m}$$

is incident normally on a lossless medium ($\varepsilon = 2\varepsilon_o$, $\mu = 8\mu_o$) in region $z \geq 0$. Determine the reflected wave $\mathbf{H}_r$, $\mathbf{E}_r$ and the transmitted wave $\mathbf{H}_t$, $\mathbf{E}_t$.

Solution:

This problem can be solved in two different ways.

Method 1: Consider the problem as illustrated in Figure 10.14. For free space,

$$\beta_1 = \frac{\omega}{c} = \frac{10^8}{3 \times 10^8} = \frac{1}{3}$$

$$\eta_1 = \eta_o = 120\pi$$

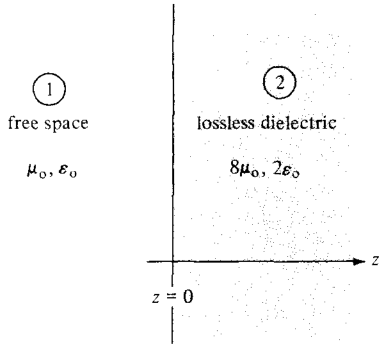

Figure 10.14 For Example 10.8.

For the lossless dielectric medium,

$$\beta_2 = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\varepsilon_o}\sqrt{\mu_r\varepsilon_r} = \frac{\omega}{c}\cdot(4) = 4\beta_1 = \frac{4}{3}$$

$$\eta_2 = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_o}{\varepsilon_o}}\sqrt{\frac{\mu_r}{\varepsilon_r}} = 2\,\eta_o$$

Given that $\mathbf{H}_i = 10\cos(10^8 t - \beta_1 z)\,\mathbf{a}_x$, we expect that

$$\mathbf{E}_i = E_{io}\cos(10^8 t - \beta_1 z)\,\mathbf{a}_{E_i}$$

where

$$\mathbf{a}_{E_i} = \mathbf{a}_{H_i} \times \mathbf{a}_{k_i} = \mathbf{a}_x \times \mathbf{a}_z = -\mathbf{a}_y$$

and

$$E_{io} = \eta_1 H_{io} = 10\,\eta_o$$

Hence,

$$\mathbf{E}_i = -10\eta_o\cos(10^8 t - \beta_1 z)\,\mathbf{a}_y \text{ mV/m}$$

Now

$$\frac{E_{ro}}{E_{io}} = \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{2\eta_o - \eta_o}{2\eta_o + \eta_o} = \frac{1}{3}$$

$$E_{ro} = \frac{1}{3}E_{io}$$

Thus

$$\mathbf{E}_r = -\frac{10}{3}\eta_o\cos\left(10^8 t + \frac{1}{3}z\right)\mathbf{a}_y \text{ mV/m}$$

from which we easily obtain $\mathbf{H}_r$ as

$$\mathbf{H}_r = -\frac{10}{3}\cos\left(10^8 t + \frac{1}{3}z\right)\mathbf{a}_x \text{ mA/m}$$

Similarly,

$$\frac{E_{to}}{E_{io}} = \tau = 1 + \Gamma = \frac{4}{3} \qquad \text{or} \qquad E_{to} = \frac{4}{3}E_{io}$$

Thus

$$\mathbf{E}_t = E_{to}\cos(10^8 t - \beta_2 z)\,\mathbf{a}_{E_t}$$

where $\mathbf{a}_{E_t} = \mathbf{a}_{E_i} = -\mathbf{a}_y$. Hence,

$$\mathbf{E}_t = -\frac{40}{3}\eta_o\cos\left(10^8 t - \frac{4}{3}z\right)\mathbf{a}_y \text{ mV/m}$$

from which we obtain

$$\mathbf{H}_t = \frac{20}{3}\cos\left(10^8 t - \frac{4}{3}z\right)\mathbf{a}_x \text{ mA/m}$$

Method 2: Alternatively, we can obtain $\mathbf{H}_r$ and $\mathbf{H}_t$ directly from $\mathbf{H}_i$ using

$$\frac{H_{ro}}{H_{io}} = -\Gamma \qquad \text{and} \qquad \frac{H_{to}}{H_{io}} = \tau\frac{\eta_1}{\eta_2}$$

Thus

$$H_{ro} = -\frac{1}{3}H_{io} = -\frac{10}{3}$$

$$H_{to} = \frac{4}{3}\frac{\eta_o}{2\eta_o}\cdot H_{io} = \frac{2}{3}H_{io} = \frac{20}{3}$$

and

$$\mathbf{H}_r = -\frac{10}{3}\cos(10^8 t + \beta_1 z)\,\mathbf{a}_x \text{ mA/m}$$

$$\mathbf{H}_t = \frac{20}{3}\cos(10^8 t - \beta_2 z)\,\mathbf{a}_x \text{ mA/m}$$

as previously obtained.

Notice that the boundary conditions at $z = 0$, namely,

$$\mathbf{E}_i(0) + \mathbf{E}_r(0) = \mathbf{E}_t(0) = -\frac{40}{3}\eta_o\cos(10^8 t)\,\mathbf{a}_y$$

and

$$\mathbf{H}_i(0) + \mathbf{H}_r(0) = \mathbf{H}_t(0) = \frac{20}{3}\cos(10^8 t)\,\mathbf{a}_x$$

are satisfied. These boundary conditions can always be used to cross-check **E** and **H**.

PRACTICE EXERCISE 10.8

A 5-GHz uniform plane wave $\mathbf{E}_{is} = 10\,e^{-j\beta z}\,\mathbf{a}_x$ V/m in free space is incident normally on a large plane, lossless dielectric slab ($z > 0$) having $\varepsilon = 4\varepsilon_o$, $\mu = \mu_o$. Find the reflected wave $\mathbf{E}_{rs}$ and the transmitted wave $\mathbf{E}_{ts}$.

Answer: $-3.333\exp(j\beta_1 z)\,\mathbf{a}_x$ V/m, $6.667\exp(-j\beta_2 z)\,\mathbf{a}_x$ V/m where $\beta_2 = 2\beta_1 = 200\pi/3$.

EXAMPLE 10.9

Given a uniform plane wave in air as

$$\mathbf{E}_i = 40\cos(\omega t - \beta z)\,\mathbf{a}_x + 30\sin(\omega t - \beta z)\,\mathbf{a}_y \text{ V/m}$$

(a) Find $\mathbf{H}_i$.

(b) If the wave encounters a perfectly conducting plate normal to the z axis at $z = 0$, find the reflected wave $\mathbf{E}_r$ and $\mathbf{H}_r$.

(c) What are the total **E** and **H** fields for $z \leq 0$?

(d) Calculate the time-average Poynting vectors for $z \leq 0$ and $z \geq 0$.

Solution:

(a) This is similar to the problem in Example 10.3. We may treat the wave as consisting of two waves $\mathbf{E}_{i1}$ and $\mathbf{E}_{i2}$, where

$$\mathbf{E}_{i1} = 40\cos(\omega t - \beta z)\,\mathbf{a}_x, \qquad \mathbf{E}_{i2} = 30\sin(\omega t - \beta z)\,\mathbf{a}_y$$

At atmospheric pressure, air has $\varepsilon_r = 1.0006 \simeq 1$. Thus air may be regarded as free space. Let $\mathbf{H}_i = \mathbf{H}_{i1} + \mathbf{H}_{i2}$.

$$\mathbf{H}_{i1} = H_{i1o}\cos(\omega t - \beta z)\,\mathbf{a}_{H_1}$$

where

$$H_{i1o} = \frac{E_{i1o}}{\eta_o} = \frac{40}{120\pi} = \frac{1}{3\pi}$$

$$\mathbf{a}_{H_1} = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$$

Hence

$$\mathbf{H}_{i1} = \frac{1}{3\pi}\cos(\omega t - \beta z)\,\mathbf{a}_y$$

Similarly,

$$\mathbf{H}_{i2} = H_{i2o}\sin(\omega t - \beta z)\,\mathbf{a}_{H_2}$$

where

$$H_{i2o} = \frac{E_{i2o}}{\eta_o} = \frac{30}{120\pi} = \frac{1}{4\pi}$$

$$\mathbf{a}_{H_2} = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_y = -\mathbf{a}_x$$

Hence

$$\mathbf{H}_{i2} = -\frac{1}{4\pi}\sin(\omega t - \beta z)\,\mathbf{a}_x$$

and

$$\begin{aligned}\mathbf{H}_i &= \mathbf{H}_{i1} + \mathbf{H}_{i2}\\ &= -\frac{1}{4\pi}\sin(\omega t - \beta z)\,\mathbf{a}_x + \frac{1}{3\pi}\cos(\omega t - \beta z)\,\mathbf{a}_y \text{ mA/m}\end{aligned}$$

This problem can also be solved using Method 2 of Example 10.3.

(b) Since medium 2 is perfectly conducting,

$$\frac{\sigma_2}{\omega\varepsilon_2} \gg 1 \quad \rightarrow \quad \eta_2 \ll \eta_1$$

that is,

$$\Gamma \simeq -1, \qquad \tau = 0$$

showing that the incident **E** and **H** fields are totally reflected.

$$E_{ro} = \Gamma\, E_{io} = -E_{io}$$

Hence,

$$\mathbf{E}_r = -40\cos(\omega t + \beta z)\,\mathbf{a}_x - 30\sin(\omega t + \beta z)\,\mathbf{a}_y \text{ V/m}$$

$\mathbf{H}_r$ can be found from $\mathbf{E}_r$ just as we did in part (a) of this example or by using Method 2 of the last example starting with $\mathbf{H}_i$. Whichever approach is taken, we obtain

$$\mathbf{H}_r = \frac{1}{3\pi}\cos(\omega t + \beta z)\,\mathbf{a}_y - \frac{1}{4\pi}\sin(\omega t + \beta z)\mathbf{a}_x \text{ A/m}$$

(c) The total fields in air

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r \qquad \text{and} \qquad \mathbf{H}_1 = \mathbf{H}_i + \mathbf{H}_r$$

can be shown to be standing wave. The total fields in the conductor are

$$\mathbf{E}_2 = \mathbf{E}_t = 0, \qquad \mathbf{H}_2 = \mathbf{H}_t = 0.$$

(d) For $z \leq 0$,

$$\begin{aligned}\mathcal{P}_{1\text{ave}} &= \frac{|\mathbf{E}_{1s}|^2}{2\eta_1}\mathbf{a}_k = \frac{1}{2\eta_o}[E_{io}^2\mathbf{a}_z - E_{ro}^2\mathbf{a}_z] \\ &= \frac{1}{240\pi}[(40^2 + 30^2)\mathbf{a}_z - (40^2 + 30^2)\mathbf{a}_z] \\ &= 0\end{aligned}$$

For $z \geq 0$,

$$\mathcal{P}_{2\text{ave}} = \frac{|\mathbf{E}_{2S}|^2}{2\eta_2}\mathbf{a}_k = \frac{E_{to}^2}{2\eta_2}\mathbf{a}_z = 0$$

because the whole incident power is reflected.

PRACTICE EXERCISE 10.9

The plane wave $\mathbf{E} = 50 \sin(\omega t - 5x)\,\mathbf{a}_y$ V/m in a lossless medium ($\mu = 4\mu_o$, $\varepsilon = \varepsilon_o$) encounters a lossy medium ($\mu = \mu_o$, $\varepsilon = 4\varepsilon_o$, $\sigma = 0.1$ mhos/m) normal to the x-axis at $x = 0$. Find

(a) Γ, τ, and s

(b) $\mathbf{E}_r$ and $\mathbf{H}_r$

(c) $\mathbf{E}_t$ and $\mathbf{H}_t$

(d) The time-average Poynting vectors in both regions

Answer: (a) $0.8186 \angle 171.1°$, $0.2295 \angle 33.56°$, 10.025, (b) $40.93 \sin(\omega t + 5x + 171.9°)\,\mathbf{a}_y$ V/m, $-54.3 \sin(\omega t + 5x + 171.9°\,\mathbf{a}_z$ mA/m, (c) $11.47\, e^{-6.021x} \sin(\omega t - 7.826x + 33.56°)\,\mathbf{a}_y$ V/m, $120.2\, e^{-6.021x} \sin(\omega t - 7.826x - 4.01°)\,\mathbf{a}_z$ mA/m, (d) $0.5469\,\mathbf{a}_x$ W/m^2, $0.5469 \exp(-12.04x)\,\mathbf{a}_x$ W/m^2.

10.9 REFLECTION OF A PLANE WAVE AT OBLIQUE INCIDENCE

We now consider a more general situation than that in Section 10.8. To simplify the analysis, we will assume that we are dealing with lossless media. (We may extend our analysis to that of lossy media by merely replacing ε by ε_c.) It can be shown (see Problems 10.14 and 10.15) that a uniform plane wave takes the general form of

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \mathbf{E}_o \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \\ &= \text{Re}\,[E_o e^{j(\mathbf{k} \cdot \mathbf{r} - \omega t)}] \end{aligned} \tag{10.93}$$

where $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ is the radius or position vector and $\mathbf{k} = k_x\mathbf{a}_x + k_y\mathbf{a}_y + k_z\mathbf{a}_z$ is the *wave number vector* or the *propagation vector;* $\mathbf{k}$ is always in the direction of wave propagation. The magnitude of $\mathbf{k}$ is related to ω according to the dispersion relation

$$k^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\varepsilon \tag{10.94}$$

Thus, for lossless media, k is essentially the same as β in the previous sections. With the general form of $\mathbf{E}$ as in eq. (10.93), Maxwell's equations reduce to

$$\mathbf{k} \times \mathbf{E} = \omega\mu\mathbf{H} \tag{10.95a}$$

$$\mathbf{k} \times \mathbf{H} = -\omega\varepsilon\mathbf{E} \tag{10.95b}$$

$$\mathbf{k} \cdot \mathbf{H} = 0 \tag{10.95c}$$

$$\mathbf{k} \cdot \mathbf{E} = 0 \tag{10.95d}$$

showing that (i) $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{k}$ are mutually orthogonal, and (ii) $\mathbf{E}$ and $\mathbf{H}$ lie on the plane

$$\mathbf{k} \cdot \mathbf{r} = k_x x + k_y y + k_z z = \text{constant}$$

From eq. (10.95a), the $\mathbf{H}$ field corresponding to the $\mathbf{E}$ field in eq. (10.93) is

$$\mathbf{H} = \frac{1}{\omega\mu}\mathbf{k} \times \mathbf{E} = \frac{\mathbf{a}_k \times \mathbf{E}}{\eta} \tag{10.96}$$

Having expressed $\mathbf{E}$ and $\mathbf{H}$ in the general form, we can now consider the oblique incidence of a uniform plane wave at a plane boundary as illustrated in Figure 10.15(a). The plane defined by the propagation vector $\mathbf{k}$ and a unit normal vector $\mathbf{a}_n$ to the boundary is called the *plane of incidence.* The angle θ_i between $\mathbf{k}$ and $\mathbf{a}_n$ is the *angle of incidence.*

Again, both the incident and the reflected waves are in medium 1 while the transmitted (or refracted wave) is in medium 2. Let

$$\mathbf{E}_i = \mathbf{E}_{io} \cos(k_{ix}x + k_{iy}y + k_{iz}z - \omega_i t) \tag{10.97a}$$

$$\mathbf{E}_r = \mathbf{E}_{ro} \cos(k_{rx}x + k_{ry}y + k_{rz}z - \omega_r t) \tag{10.97b}$$

$$\mathbf{E}_t = \mathbf{E}_{to} \cos(k_{tx}x + k_{ty}y + k_{tz}z - \omega_t t) \tag{10.97c}$$

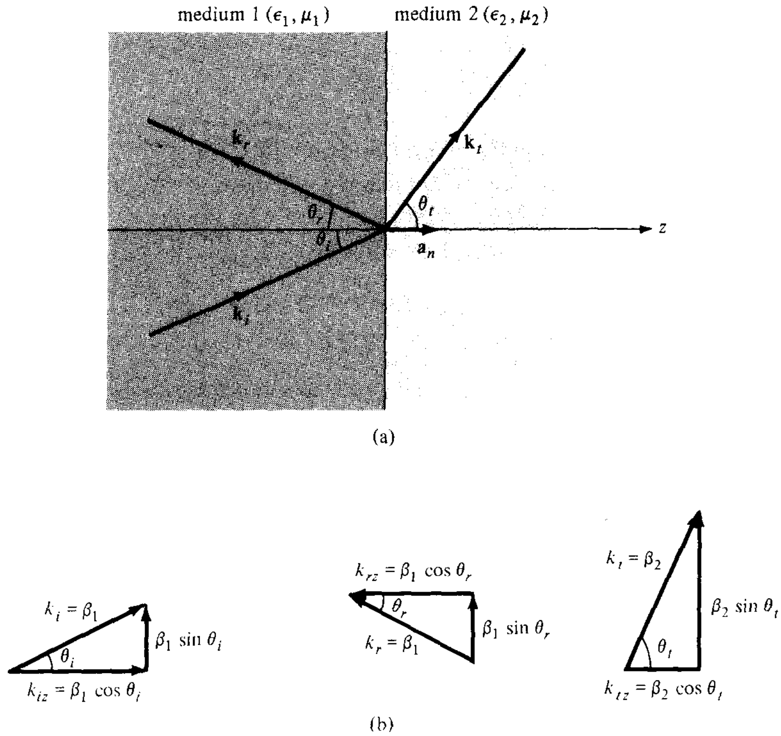

Figure 10.15 Oblique incidence of a plane wave: **(a)** illustration of θ_i, θ_r, and θ_t; **(b)** illustration of the normal and tangential components of **k**.

where k_i, k_r, and k_t with their normal and tangential components are shown in Figure 10.15(b). Since the tangential component of **E** must be continuous at the boundary $z = 0$,

$$\mathbf{E}_i(z = 0) + \mathbf{E}_r(z = 0) = \mathbf{E}_t(z = 0) \tag{10.98}$$

The only way this boundary condition will be satisfied by the waves in eq. (10.97) for all x and y is that

1. $\omega_i = \omega_r = \omega_t = \omega$
2. $k_{ix} = k_{rx} = k_{tx} = k_x$
3. $k_{iy} = k_{ry} = k_{ty} = k_y$

Condition 1 implies that the frequency is unchanged. Conditions 2 and 3 require that the tangential components of the propagation vectors be continuous (called the *phase matching conditions*). This means that the propagation vectors $\mathbf{k}_i$, $\mathbf{k}_t$, and $\mathbf{k}_r$ must all lie in the plane of incidence. Thus, by conditions 2 and 3,

$$k_i \sin\theta_i = k_r \sin\theta_r \tag{10.99}$$

$$k_i \sin\theta_i = k_t \sin\theta_t \tag{10.100}$$

where θ_r is *the angle of reflection* and θ_t is the *angle of transmission.* But for lossless media,

$$k_i = k_r = \beta_1 = \omega\sqrt{\mu_1\varepsilon_1} \tag{10.101a}$$

$$k_t = \beta_2 = \omega\sqrt{\mu_2\varepsilon_2} \tag{10.101b}$$

From eqs. (10.99) and (10.101a), it is clear that

$$\boxed{\theta_r = \theta_i} \tag{10.102}$$

so that the angle of reflection θ_r equals the angle of incidence θ_i as in optics. Also from eqs. (10.100) and (10.101),

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{k_i}{k_t} = \frac{\mu_2}{\mu_1} = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}} \tag{10.103}$$

where $u = \omega/k$ is the phase velocity. Equation (10.103) is the well-known *Snell's law,* which can be written as

$$\boxed{n_1 \sin\theta_i = n_2 \sin\theta_t} \tag{10.104}$$

where $n_1 = c\sqrt{\mu_1\varepsilon_1} = c/u_1$ and $n_2 = c\sqrt{\mu_2\varepsilon_2} = c/u_2$ are the *refractive indices* of the media.

Based on these general preliminaries on oblique incidence, we will now specifically consider two special cases: one with the **E** field perpendicular to the plane of incidence, the other with the **E** field parallel to it. Any other polarization may be considered as a linear combination of these two cases.

A. Parallel Polarization

This case is illustrated in Figure 10.16 where the **E** field lies in the *xz*-plane, the plane of incidence. In medium 1, we have both incident and reflected fields given by

$$\mathbf{E}_{is} = E_{io}(\cos\theta_i\,\mathbf{a}_x - \sin\theta_i\,\mathbf{a}_z)\,e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \tag{10.105a}$$

$$\mathbf{H}_{is} = \frac{E_{io}}{\eta_1}\,e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\,\mathbf{a}_y \tag{10.105b}$$

$$\mathbf{E}_{rs} = E_{ro}(\cos\theta_r\,\mathbf{a}_x + \sin\theta_r\,\mathbf{a}_z)\,e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \tag{10.106a}$$

$$\mathbf{H}_{rs} = -\frac{E_{ro}}{\eta_1}\,e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}\,\mathbf{a}_y \tag{10.106b}$$

where $\beta_1 = \omega\sqrt{\mu_1\varepsilon_1}$. Notice carefully how we arrive at each field component. The trick in deriving the components is to first get the polarization vector **k** as shown in Figure 10.15(b) for incident, reflected, and transmitted waves. Once **k** is known, we

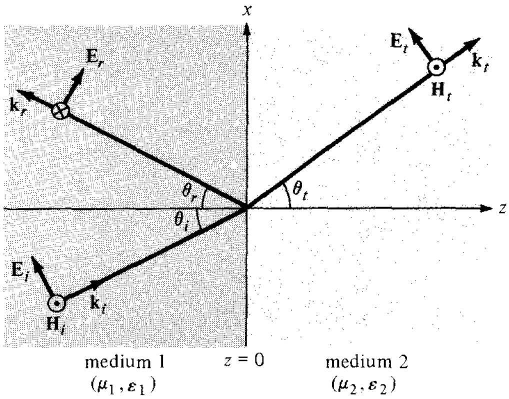

Figure 10.16 Oblique incidence with **E** parallel to the plane of incidence.

define $\mathbf{E}_s$ such that $\nabla \cdot \mathbf{E}_s = 0$ or $\mathbf{k} \cdot \mathbf{E}_s = 0$ and then $\mathbf{H}_s$ is obtained from $\mathbf{H}_s = \dfrac{\mathbf{k}}{\omega\mu} \times \mathbf{E}_s = \mathbf{a}_k \times \dfrac{\mathbf{E}}{\eta}$.

The transmitted fields exist in medium 2 and are given by

$$\mathbf{E}_{ts} = E_{to}(\cos\theta_t\,\mathbf{a}_x - \sin\theta_t\,\mathbf{a}_z)\,e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \tag{10.107a}$$

$$\mathbf{H}_{ts} = \frac{E_{to}}{\eta_2}\,e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}\,\mathbf{a}_y \tag{10.107b}$$

where $\beta_2 = \omega\sqrt{\mu_2\varepsilon_2}$. Should our assumption about the relative directions in eqs. (10.105) to (10.107) be wrong, the final result will show us by means of its sign.

Requiring that $\theta_r = \theta_i$ and that the tangential components of **E** and **H** be continuous at the boundary $z = 0$, we obtain

$$(E_{io} + E_{ro})\cos\theta_i = E_{to}\cos\theta_t \tag{10.108a}$$

$$\frac{1}{\eta_1}(E_{io} - E_{ro}) = \frac{1}{\eta_2}E_{to} \tag{10.108b}$$

Expressing E_{ro} and E_{to} in terms of E_{io}, we obtain

$$\boxed{\Gamma_\| = \frac{E_{ro}}{E_{io}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}} \tag{10.109a}$$

or

$$E_{ro} = \Gamma_\| E_{io} \tag{10.109b}$$

and

$$\boxed{\tau_\| = \frac{E_{to}}{E_{io}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}} \tag{10.110a}$$

or

$$E_{to} = \tau_{\|} E_{io} \tag{10.110b}$$

Equations (10.109) and (10.110) are called *Fresnel's equations.* Note that the equations reduce to eqs. (10.81) and (10.82) when $\theta_i = \theta_t = 0$ as expected. Since θ_i and θ_t are related according to Snell's law of eq. (10.103), eqs. (10.109) and (10.110) can be written in terms of θ_i by substituting

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - (u_2/u_1)^2 \sin^2\theta_i} \tag{10.111}$$

From eqs. (10.109) and (10.110), it is easily shown that

$$1 + \Gamma_{\|} = \tau_{\|}\left(\frac{\cos\theta_t}{\cos\theta_i}\right) \tag{10.112}$$

From eq. (10.109a), it is evident that it is possible that $\Gamma_{\|} = 0$ because the numerator is the difference of two terms. Under this condition, there is no reflection ($E_{ro} = 0$) and the incident angle at which this takes place is called the *Brewster angle* $\theta_{B_{\|}}$. The Brewster angle is also known as the *polarizing angle* because an arbitrarily polarized incident wave will be reflected with only the component of **E** perpendicular to the plane of incidence. The Brewster effect is utilized in a laser tube where quartz windows are set at the Brewster angle to control polarization of emitted light. The Brewster angle is obtained by setting $\theta_i = \theta_{B_{\|}}$ when $\Gamma_{\|} = 0$ in eq. (10.109), that is,

$$\eta_2 \cos\theta_t = \eta_1 \cos\theta_{B_{\|}}$$

or

$$\eta_2^2 (1 - \sin^2\theta_t) = \eta_1^2 (1 - \sin^2\theta_{B_{\|}})$$

Introducing eq. (10.103) or (10.104) gives

$$\boxed{\sin^2\theta_{B_{\|}} = \frac{1 - \mu_2\varepsilon_1/\mu_1\varepsilon_2}{1 - (\varepsilon_1/\varepsilon_2)^2}} \tag{10.113}$$

It is of practical value to consider the case when the dielectric media are not only lossless but nonmagnetic as well—that is, $\mu_1 = \mu_2 = \mu_o$. For this situation, eq. (10.113) becomes

$$\sin^2\theta_{B_{\|}} = \frac{1}{1 + \varepsilon_1/\varepsilon_2} \rightarrow \sin\theta_{B_{\|}} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}}$$

or

$$\tan\theta_{B_{\|}} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1} \tag{10.114}$$

showing that there is a Brewster angle for any combination of ε_1 and ε_2.

B. Perpendicular Polarization

In this case, the **E** field is perpendicular to the plane of incidence (the xz-plane) as shown in Figure 10.17. This may also be viewed as the case where **H** field is parallel to the plane of incidence. The incident and reflected fields in medium 1 are given by

$$\mathbf{E}_{is} = E_{io}e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\,\mathbf{a}_y \tag{10.115a}$$

$$\mathbf{H}_{is} = \frac{E_{io}}{\eta_1}(-\cos\theta_i\,\mathbf{a}_x + \sin\theta_i\,\mathbf{a}_z)\,e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \tag{10.115b}$$

$$\mathbf{E}_{rs} = E_{ro}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}\,\mathbf{a}_y \tag{10.116a}$$

$$\mathbf{H}_{rs} = \frac{E_{ro}}{\eta_1}(\cos\theta_r\,\mathbf{a}_x + \sin\theta_r\,\mathbf{a}_z)\,e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \tag{10.116b}$$

while the transmitted fields in medium 2 are given by

$$\mathbf{E}_{ts} = E_{to}e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}\,\mathbf{a}_y \tag{10.117a}$$

$$\mathbf{H}_{ts} = \frac{E_{to}}{\eta^2}(-\cos\theta_t\,\mathbf{a}_x + \sin\theta_t\,\mathbf{a}_z)\,e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \tag{10.117b}$$

Notice that in defining the field components in eqs. (10.115) to (10.117), Maxwell's equations (10.95) are always satisfied. Again, requiring that the tangential components of **E** and **H** be continuous at $z = 0$ and setting θ_r equal to θ_i, we get

$$E_{io} + E_{ro} = E_{to} \tag{10.118a}$$

$$\frac{1}{\eta_1}(E_{io} - E_{ro})\cos\theta_i = \frac{1}{\eta_2}E_{to}\cos\theta_t \tag{10.118b}$$

Expressing E_{ro} and E_{to} in terms of E_{io} leads to

$$\boxed{\Gamma_\perp = \frac{E_{ro}}{E_{io}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}} \tag{10.119a}$$

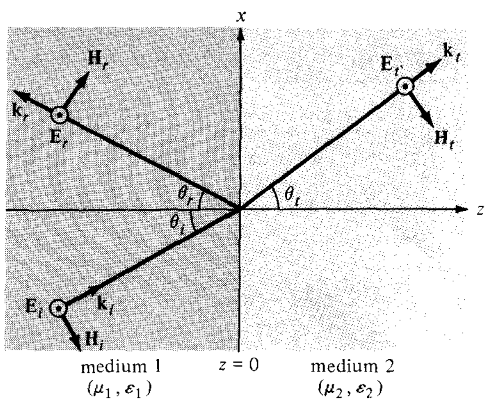

Figure 10.17 Oblique incidence with **E** perpendicular to the plane of incidence.

or

$$E_{ro} = \Gamma_\perp E_{io} \tag{10.119b}$$

and

$$\boxed{\tau_\perp = \frac{E_{to}}{E_{io}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}} \tag{10.120a}$$

or

$$E_{to} = \tau_\perp E_{io} \tag{10.120b}$$

which are the *Fresnel's equations* for perpendicular polarization. From eqs. (10.119) and (10.120), it is easy to show that

$$1 + \Gamma_\perp = \tau_\perp \tag{10.121}$$

which is similar to eq. (10.83) for normal incidence. Also, when $\theta_i = \theta_t = 0$, eqs. (10.119) and (10.120) become eqs. (10.81) and (10.82) as they should.

For no reflection, $\Gamma_\perp = 0$ (or $E_r = 0$). This is the same as the case of total transmission ($\tau_\perp = 1$). By replacing θ_i with the corresponding Brewster angle $\theta_{B_\perp}$, we obtain

$$\eta_2 \cos\theta_{B_\perp} = \eta_1 \cos\theta_t$$

or

$$\eta_2^2 (1 - \sin^2\theta_{B_\perp}) = \eta_1^2 (1 - \sin^2\theta_t)$$

Incorporating eq. (10.104) yields

$$\sin^2\theta_{B_\perp} = \frac{1 - \mu_1\varepsilon_2/\mu_2\varepsilon_1}{1 - (\mu_1/\mu_2)^2} \tag{10.122}$$

Note that for nonmagnetic media ($\mu_1 = \mu_2 = \mu_o$), $\sin^2\theta_{B_\perp} \to \infty$ in eq. (10.122), so $\theta_{B_\perp}$ does not exist because the sine of an angle is never greater than unity. Also if $\mu_1 \neq \mu_2$ and $\varepsilon_1 = \varepsilon_2$, eq. (10.122) reduces to

$$\sin\theta_{B_\perp} = \sqrt{\frac{\mu_2}{\mu_1 + \mu_2}}$$

or

$$\tan\theta_{B_\perp} = \sqrt{\frac{\mu_2}{\mu_1}} \tag{10.123}$$

Although this situation is theoretically possible, it is rare in practice.

EXAMPLE 10.10 An EM wave travels in free space with the electric field component

$$\mathbf{E}_s = 100\, e^{j(0.866y+0.5z)}\, \mathbf{a}_x \text{ V/m}$$

Determine

(a) ω and λ
(b) The magnetic field component
(c) The time average power in the wave

Solution:

(a) Comparing the given **E** with

$$\mathbf{E}_s = \mathbf{E}_o\, e^{j\mathbf{k}\cdot\mathbf{r}} = E_o e^{j(k_x x + k_y y + k_z z)}\, \mathbf{a}_x$$

it is clear that

$$k_x = 0, \qquad k_y = 0.866, \qquad k_z = 0.5$$

Thus

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \sqrt{(0.866)^2 + (0.5)^2} = 1$$

But in free space,

$$k = \beta = \omega\sqrt{\mu_o \varepsilon_o} = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

Hence,

$$\omega = kc = 3 \times 10^8 \text{ rad/s}$$

$$\lambda = \frac{2\pi}{k} = 2\pi = 6.283 \text{ m}$$

(b) From eq. (10.96), the corresponding magnetic field is given by

$$\begin{aligned} \mathbf{H}_s &= \frac{1}{\mu\omega} \mathbf{k} \times \mathbf{E}_s \\ &= \frac{(0.866\mathbf{a}_y + 0.5\mathbf{a}_z)}{4\pi \times 10^{-7} \times 3 \times 10^8} \times 100\, \mathbf{a}_x e^{j\mathbf{k}\cdot\mathbf{r}} \end{aligned}$$

or

$$\mathbf{H}_s = (1.33\, \mathbf{a}_y - 2.3\, \mathbf{a}_z)\, e^{j(0.866y+0.5z)} \text{ mA/m}$$

(c) The time average power is

$$\begin{aligned} \mathcal{P}_{\text{ave}} &= \frac{1}{2} \text{Re}\,(\mathbf{E}_s \times \mathbf{H}_s^*) = \frac{E_o^2}{2\eta}\, \mathbf{a}_k \\ &= \frac{(100)^2}{2(120\pi)} (0.866\, \mathbf{a}_y + 0.5\, \mathbf{a}_z) \\ &= 11.49\, \mathbf{a}_y + 6.631\, \mathbf{a}_z \text{ W/m}^2 \end{aligned}$$

PRACTICE EXERCISE 10.10

Rework Example 10.10 if

$$\mathbf{E} = (10\,\mathbf{a}_y + 5\mathbf{a}_z)\cos(\omega t + 2y - 4z) \text{ V/m}$$

in free space.

Answer: (a) 1.342×10^9 rad/s, 1.405 m, (b) $-29.66 \cos(1.342 \times 10^9 t + 2y - 4z)\,\mathbf{a}_x$ mA/m, (c) $-0.07415\,\mathbf{a}_y + 0.1489\,\mathbf{a}_z$ W/m^2.

EXAMPLE 10.11

A uniform plane wave in air with

$$\mathbf{E} = 8\cos(\omega t - 4x - 3z)\,\mathbf{a}_y \text{ V/m}$$

is incident on a dielectric slab ($z \geq 0$) with $\mu_r = 1.0$, $\varepsilon_r = 2.5$, $\sigma = 0$. Find

(a) The polarization of the wave
(b) The angle of incidence
(c) The reflected **E** field
(d) The transmitted **H** field

Solution:

(a) From the incident **E** field, it is evident that the propagation vector is

$$\mathbf{k}_i = 4\mathbf{a}_x + 3\mathbf{a}_z \rightarrow k_i = 5 = \omega\sqrt{\mu_o\varepsilon_o} = \frac{\omega}{c}$$

Hence,

$$\omega = 5c = 15 \times 10^8 \text{ rad/s.}$$

A unit vector normal to the interface ($z = 0$) is $\mathbf{a}_z$. The plane containing **k** and $\mathbf{a}_z$ is y = constant, which is the xz-plane, the plane of incidence. Since $\mathbf{E}_i$ is normal to this plane, we have perpendicular polarization (similar to Figure 10.17).

(b) The propagation vectors are illustrated in Figure 10.18 where it is clear that

$$\tan\theta_i = \frac{k_{ix}}{k_{iz}} = \frac{4}{3} \rightarrow \theta_i = 53.13°$$

Alternatively, without Figure 10.18, we can obtain θ_i from the fact that θ_i is the angle between **k** and $\mathbf{a}_n$, that is,

$$\cos\theta_i = \mathbf{a}_k \cdot \mathbf{a}_n = \left(\frac{4\mathbf{a}_x + 3\mathbf{a}_z}{5}\right) \cdot \mathbf{a}_z = \frac{3}{5}$$

or

$$\theta_i = 53.13°$$

(c) An easy way to find $\mathbf{E}_r$ is to use eq. (10.116a) because we have noticed that this problem is similar to that considered in Section 10.9(b). Suppose we are not aware of this. Let

$$\mathbf{E}_r = E_{ro} \cos(\omega t - \mathbf{k}_r \cdot \mathbf{r})\, \mathbf{a}_y$$

which is similar to form to the given $\mathbf{E}_i$. The unit vector $\mathbf{a}_y$ is chosen in view of the fact that the tangential component of $\mathbf{E}$ must be continuous at the interface. From Figure 10.18,

$$\mathbf{k}_r = k_{rx}\, \mathbf{a}_x - k_{rz}\, \mathbf{a}_z$$

where

$$k_{rx} = k_r \sin\theta_r, \qquad k_{rz} = k_r \cos\theta_r$$

But $\theta_r = \theta_i$ and $k_r = k_i = 5$ because both k_r and k_i are in the same medium. Hence,

$$k_r = 4\mathbf{a}_x - 3\mathbf{a}_z$$

To find E_{ro}, we need θ_t. From Snell's law

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i = \frac{c\sqrt{\mu_1\varepsilon_1}}{c\sqrt{\mu_2\varepsilon_2}}\sin\theta_i$$
$$= \frac{\sin 53.13°}{\sqrt{2.5}}$$

or

$$\theta_t = 30.39°$$

$$\Gamma_\perp = \frac{E_{ro}}{E_{io}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

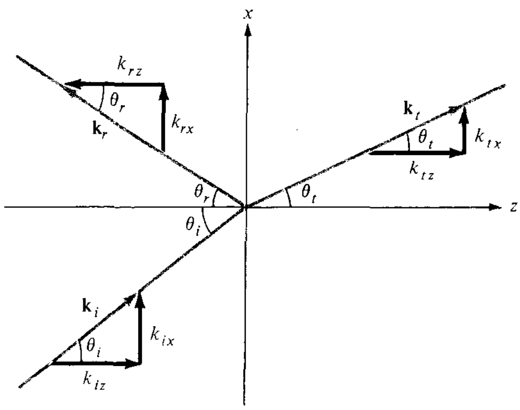

Figure 10.18 Propagation vectors of Example 10.11.

where $\eta_1 = \eta_o = 377, \qquad \eta_2 = \sqrt{\frac{\mu_o \mu_{r_2}}{\varepsilon_o \varepsilon_{r_2}}} = \frac{377}{\sqrt{2.5}} = 238.4$

$$\Gamma_\perp = \frac{238.4 \cos 35.13° - 377 \cos 30.39°}{238.4 \cos 53.13° + 377 \cos 30.39°} = -0.389$$

Hence,

$$E_{ro} = \Gamma_\perp E_{io} = -0.389\,(8) = -3.112$$

and

$$\mathbf{E}_r = -3.112 \cos (15 \times 10^8 t - 4x + 3z)\, \mathbf{a}_y \text{ V/m}$$

(d) Similarly, let the transmitted electric field be

$$\mathbf{E}_t = E_{to} \cos (\omega t - \mathbf{k}_t \cdot \mathbf{r})\, \mathbf{a}_y$$

where

$$\begin{aligned} k_t = \beta_2 &= \omega \sqrt{\mu_2 \varepsilon_2} = \frac{\omega}{c} \sqrt{\mu_{r_2} \varepsilon_{r_2}} \\ &= \frac{15 \times 10^8}{3 \times 10^8} \sqrt{1 \times 2.5} = 7.906 \end{aligned}$$

From Figure 10.18,

$$k_{tx} = k_t \sin \theta_t = 4$$

$$k_{tz} = k_t \cos \theta_t = 6.819$$

or

$$\mathbf{k}_t = 4\mathbf{a}_x + 6.819\, \mathbf{a}_z$$

Notice that $k_{ix} = k_{rx} = k_{tx}$ as expected.

$$\begin{aligned} \tau_\perp = \frac{E_{to}}{E_{io}} &= \frac{2\, \eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} \\ &= \frac{2 \times 238.4 \cos 53.13°}{238.4 \cos 53.13° + 377 \cos 30.39°} \\ &= 0.611 \end{aligned}$$

The same result could be obtained from the relation $\tau_\perp = 1 + \Gamma_\perp$. Hence,

$$E_{to} = \tau_\perp E_{io} = 0.611 \times 8 = 4.888$$

$$\mathbf{E}_t = 4.888 \cos (15 \times 10^8 t - 4x - 6.819z)\, \mathbf{a}_y$$

From $\mathbf{E}_t$, $\mathbf{H}_t$ is easily obtained as

$$\mathbf{H}_t = \frac{1}{\mu_2 \omega} \mathbf{k}_t \times \mathbf{E}_t = \frac{\mathbf{a}_{k_t} \times \mathbf{E}_t}{\eta_2}$$

$$= \frac{4\mathbf{a}_x + 6.819\,\mathbf{a}_z}{7.906\,(238.4)} \times 4.888\,\mathbf{a}_y \cos(\omega t - \mathbf{k} \cdot \mathbf{r})$$

$$\mathbf{H}_t = (-17.69\,\mathbf{a}_x + 10.37\,\mathbf{a}_z) \cos(15 \times 10^8 t - 4x - 6.819z) \text{ mA/m.}$$

PRACTICE EXERCISE 10.11

If the plane wave of Practice Exercise 10.10 is incident on a dielectric medium having $\sigma = 0$, $\varepsilon = 4\varepsilon_o$, $\mu = \mu_o$ and occupying $z \geq 0$, calculate

(a) The angles of incidence, reflection, and transmission

(b) The reflection and transmission coefficients

(c) The total **E** field in free space

(d) The total **E** field in the dielectric

(e) The Brewster angle.

Answer: (a) 26.56°, 26.56°, 12.92°, (b) −0.295, 0.647, (c) $(10\,\mathbf{a}_y + 5\mathbf{a}_z)\cos(\omega t + 2y - 4z) + (-2.946\mathbf{a}_y + 1.473\mathbf{a}_z)\cos(\omega t + 2y + 4z)$ V/m, (d) $(7.055\mathbf{a}_y + 1.618\mathbf{a}_z)\cos(\omega t + 2y - 8.718z)$ V/m, (e) 63.43°.

SUMMARY

1. The wave equation is of the form

$$\frac{\partial^2 \Phi}{\partial t^2} - u^2 \frac{\partial^2 \Phi}{\partial z^2} = 0$$

with the solution

$$\Phi = A \sin(\omega t - \beta z)$$

where u = wave velocity, A = wave amplitude, ω = angular frequency ($=2\pi f$), and β = phase constant. Also, $\beta = \omega/u = 2\pi/\lambda$ or $u = f\lambda = \lambda/T$, where λ = wavelength and T = period.

2. In a lossy, charge-free medium, the wave equation based on Maxwell's equations is of the form

$$\nabla^2 \mathbf{A}_s - \gamma^2 \mathbf{A}_s = 0$$

where $\mathbf{A}_s$ is either $\mathbf{E}_s$ or $\mathbf{H}_s$ and $\gamma = \alpha + j\beta$ is the propagation constant. If we assume $\mathbf{E}_s = E_{xs}(z)\,\mathbf{a}_x$, we obtain EM waves of the form

$$\mathbf{E}(z, t) = E_o e^{-\alpha z} \cos(\omega t - \beta z)\,\mathbf{a}_x$$

$$\mathbf{H}(z, t) = H_o e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\,\mathbf{a}_y$$

where α = attenuation constant, β = phase constant, $\eta = |\eta|\underline{/\theta_\eta}$ = intrinsic impedance of the medium. The reciprocal of α is the skin depth ($\delta = 1/\alpha$). The relationship between β, ω, and λ as stated above remain valid for EM waves.

3. Wave propagation in other types of media can be derived from that for lossy media as special cases. For free space, set $\sigma = 0$, $\varepsilon = \varepsilon_o$, $\mu = \mu_o$; for lossless dielectric media, set $\sigma = 0$, $\varepsilon = \varepsilon_o\varepsilon_r$, and $\mu = \mu_o\mu_r$; and for good conductors, set $\sigma \simeq \infty$, $\varepsilon = \varepsilon_o$, $\mu = \mu_o$, or $\sigma/\omega\varepsilon \to 0$.
4. A medium is classified as lossy dielectric, lossless dielectric or good conductor depending on its loss tangent given by

$$\tan\theta = \frac{|\mathbf{J}_s|}{|\mathbf{J}_{d_s}|} = \frac{\sigma}{\omega\varepsilon} = \frac{\varepsilon''}{\varepsilon'}$$

where $\varepsilon_c = \varepsilon' - j\varepsilon''$ is the complex permittivity of the medium. For lossless dielectrics $\tan\theta \ll 1$, for good conductors $\tan\theta \gg 1$, and for lossy dielectrics $\tan\theta$ is of the order of unity.

5. In a good conductor, the fields tend to concentrate within the initial distance δ from the conductor surface. This phenomenon is called skin effect. For a conductor of width w and length ℓ, the effective or ac resistance is

$$R_{ac} = \frac{\ell}{\sigma w \delta}$$

where δ is the skin depth.

6. The Poynting vector, $\mathcal{P}$, is the power-flow vector whose direction is the same as the direction of wave propagation and magnitude the same as the amount of power flowing through a unit area normal to its direction.

$$\mathcal{P} = \mathbf{E} \times \mathbf{H}, \qquad \mathcal{P}_{ave} = 1/2\ \mathrm{Re}\,(\mathbf{E}_s \times \mathbf{H}_s^*)$$

7. If a plane wave is incident normally from medium 1 to medium 2, the reflection coefficient Γ and transmission coefficient τ are given by

$$\Gamma = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}, \qquad \tau = \frac{E_{to}}{E_{io}} = 1 + \Gamma$$

The standing wave ratio, s, is defined as

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

8. For oblique incidence from lossless medium 1 to lossless medium 2, we have the Fresnel coefficients as

$$\Gamma_{\|} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}, \qquad \tau_{\|} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

for parallel polarization and

$$\Gamma_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \qquad \tau_\perp = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

for perpendicular polarization. As in optics,

$$\theta_r = \theta_i$$

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{\beta_1}{\beta_2} = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}$$

Total transmission or no reflection ($\Gamma = 0$) occurs when the angle of incidence θ_i is equal to the Brewster angle.

REVIEW QUESTIONS

10.1 Which of these is not a correct form of the wave $E_x = \cos(\omega t - \beta z)$?

(a) $\cos(\beta z - \omega t)$
(b) $\sin(\beta z - \omega t - \pi/2)$
(c) $\cos\left(\frac{2\pi t}{T} - \frac{2\pi z}{\lambda}\right)$
(d) $\mathrm{Re}\,(e^{j(\omega t - \beta z)})$
(e) $\cos\beta(z - ut)$

10.2 Identify which of these functions do not satisfy the wave equation:

(a) $50e^{j\omega(t-3z)}$
(b) $\sin\omega\,(10z + 5t)$
(c) $(x + 2t)^2$
(d) $\cos^2(y + 5t)$
(e) $\sin x \cos t$
(f) $\cos(5y + 2x)$

10.3 Which of the following statements is not true of waves in general?

(a) It may be a function of time only.
(b) It may be sinusoidal or cosinusoidal.
(c) It must be a function of time and space.
(d) For practical reasons, it must be finite in extent.

10.4 The electric field component of a wave in free space is given by $\mathbf{E} = 10\cos(10^7 t + kz)\,\mathbf{a}_y$ V/m. It can be inferred that

(a) The wave propagates along $\mathbf{a}_y$.
(b) The wavelength $\lambda = 188.5$ m.

(c) The wave amplitude is 10 V/m.
(d) The wave number $k = 0.33$ rad/m.
(e) The wave attenuates as it travels.

10.5 Given that $\mathbf{H} = 0.5\, e^{-0.1x} \sin(10^6 t - 2x)\, \mathbf{a}_z$ A/m, which of these statements are incorrect?

(a) $\alpha = 0.1$ Np/m
(b) $\beta = -2$ rad/m
(c) $\omega = 10^6$ rad/s
(d) The wave travels along $\mathbf{a}_x$.
(e) The wave is polarized in the z-direction.
(f) The period of the wave is 1 μs.

10.6 What is the major factor for determining whether a medium is free space, lossless dielectric, lossy dielectric, or good conductor?

(a) Attenuation constant
(b) Constitutive parameters (σ, ε, μ)
(c) Loss tangent
(d) Reflection coefficient

10.7 In a certain medium, $\mathbf{E} = 10 \cos(10^8 t - 3y)\, \mathbf{a}_x$ V/m. What type of medium is it?

(a) Free space
(b) Perfect dielectric
(c) Lossless dielectric
(d) Perfect conductor

10.8 Electromagnetic waves travel faster in conductors than in dielectrics.

(a) True
(b) False

10.9 In a good conductor, **E** and **H** are in time phase.

(a) True
(b) False

10.10 The Poynting vector physically denotes the power density leaving or entering a given volume in a time-varying field.

(a) True
(b) False

Answers: 10.1b, 10.2d,f, 10.3a, 10.4b,c, 10.5b,f, 10.6c, 10.7c, 10.8b, 10.9b, 10.10a.

PROBLEMS

10.1 An EM wave propagating in a certain medium is described by

$$\mathbf{E} = 25 \sin (2\pi \times 10^6 t - 6x)\, \mathbf{a}_z \text{ V/m}$$

(a) Determine the direction of wave propagation.
(b) Compute the period T, the wavelength λ, and the velocity u.
(c) Sketch the wave at $t = 0, T/8, T/4, T/2$.

10.2 (a) Derive eqs. (10.23) and (10.24) from eqs. (10.18) and (10.20).
(b) Using eq. (10.29) in conjunction with Maxwell's equations, show that

$$\eta = \frac{j\omega\mu}{\gamma}$$

(c) From part (b), derive eqs. (10.32) and (10.33).

10.3 At 50 MHz, a lossy dielectric material is characterized by $\varepsilon = 3.6\varepsilon_o$, $\mu = 2.1\mu_o$, and $\sigma = 0.08$ S/m. If $\mathbf{E}_s = 6e^{-\gamma x}\, \mathbf{a}_z$ V/m, compute: (a) γ, (b) λ, (c) u, (d) η, (e) $\mathbf{H}_s$.

10.4 A lossy material has $\mu = 5\mu_o$, $\varepsilon = 2\varepsilon_o$. If at 5 MHz, the phase constant is 10 rad/m, calculate

(a) The loss tangent
(b) The conductivity of the material
(c) The complex permittivity
(d) The attenuation constant
(e) The intrinsic impedance

***10.5** A nonmagnetic medium has an intrinsic impedance $240 \angle 30^\circ$ Ω. Find its

(a) Loss tangent
(b) Dielectric constant
(c) Complex permittivity
(d) Attenuation constant at 1 MHz

10.6 The amplitude of a wave traveling through a lossy nonmagnetic medium reduces by 18% every meter. If the wave operates at 10 MHz and the electric field leads the magnetic field by 24°, calculate: (a) the propagation constant, (b) the wavelength, (c) the skin depth, (d) the conductivity of the medium.

10.7 Sea water plays a vital role in the study of submarine communications. Assuming that for sea water, $\sigma = 4$ S/m, $\varepsilon_r = 80$, $\mu_r = 1$, and $f = 100$ MHz, calculate: (a) the phase velocity, (b) the wavelength, (c) the skin depth, (d) the intrinsic impedance.

10.8 In a certain medium with $\mu = \mu_o$, $\varepsilon = 4\varepsilon_o$,

$$\mathbf{H} = 12e^{-0.1y} \sin (\pi \times 10^8 t - \beta y)\, \mathbf{a}_x \text{ A/m}$$

find: (a) the wave period T, (b) the wavelength λ, (c) the electric field $\mathbf{E}$, (d) the phase difference between $\mathbf{E}$ and $\mathbf{H}$.

10.9 In a medium,

$$\mathbf{E} = 16e^{-0.05x} \sin(2 \times 10^8 t - 2x)\, \mathbf{a}_z \text{ V/m}$$

find: (a) the propagation constant, (b) the wavelength, (c) the speed of the wave, (d) the skin depth.

10.10 A uniform wave in air has

$$\mathbf{E} = 10 \cos(2\pi \times 10^6 t - \beta z)\, \mathbf{a}_y$$

(a) Calculate β and λ.

(b) Sketch the wave at $z = 0, \lambda/4$.

(c) Find **H**.

10.11 The magnetic field component of an EM wave propagating through a nonmagnetic medium ($\mu = \mu_o$) is

$$\mathbf{H} = 25 \sin(2 \times 10^8 t + 6x)\, \mathbf{a}_y \text{ mA/m}$$

Determine:

(a) The direction of wave propagation.

(b) The permittivity of the medium.

(c) The electric field intensity.

10.12 If $\mathbf{H} = 10 \sin(\omega t - 4z)\mathbf{a}_x$ mA/m in a material for which $\sigma = 0$, $\mu = \mu_o$, $\varepsilon = 4\varepsilon_o$, calculate ω, λ, and $\mathbf{J}_d$.

10.13 A manufacturer produces a ferrite material with $\mu = 750\mu_o$, $\varepsilon = 5\varepsilon_o$, and $\sigma = 10^{-6}$ S/m at 10 MHz.

(a) Would you classify the material as lossless, lossy, or conducting?

(b) Calculate β and λ.

(c) Determine the phase difference between two points separated by 2 m.

(d) Find the intrinsic impedance.

***10.14** By assuming the time-dependent fields $\mathbf{E} = \mathbf{E}_o e^{j(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ and $\mathbf{H} = \mathbf{H}_o e^{j(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ where $\mathbf{k} = k_x\mathbf{a}_x + k_y\mathbf{a}_y + k_z\mathbf{a}_z$ is the wave number vector and $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ is the radius vector, show that $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ can be expressed as $\mathbf{k} \times \mathbf{E} = \mu\omega\mathbf{H}$ and deduce $\mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_H$.

10.15 Assume the same fields as in Problem 10.14 and show that Maxwell's equations in a source-free region can be written as

$$\mathbf{k} \cdot \mathbf{E} = 0$$

$$\mathbf{k} \cdot \mathbf{H} = 0$$

$$\mathbf{k} \times \mathbf{E} = \omega\mu\mathbf{H}$$

$$\mathbf{k} \times \mathbf{H} = -\omega\varepsilon\mathbf{E}$$

From these equations deduce

$$\mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_H \qquad \text{and} \qquad \mathbf{a}_k \times \mathbf{a}_H = -\mathbf{a}_E$$

10.16 The magnetic field component of a plane wave in a lossless dielectric is

$$\mathbf{H} = 30 \sin (2\pi \times 10^8 t - 5x)\, \mathbf{a}_z \text{ mA/m}$$

(a) If $\mu_r = 1$, find ε_r.
(b) Calculate the wavelength and wave velocity.
(c) Determine the wave impedance.
(d) Determine the polarization of the wave.
(e) Find the corresponding electric field component.
(f) Find the displacement current density.

10.17 In a nonmagnetic medium,

$$\mathbf{E} = 50 \cos (10^9 t - 8x)\, \mathbf{a}_y + 40 \sin (10^9 t - 8x)\, \mathbf{a}_z \text{ V/m}$$

find the dielectric constant ε_r and the corresponding **H**.

10.18 In a certain medium

$$\mathbf{E} = 10 \cos (2\pi \times 10^7 t - \beta x)(\mathbf{a}_y + \mathbf{a}_z) \text{ V/m}$$

If $\mu = 50\mu_o$, $\varepsilon = 2\varepsilon_o$, and $\sigma = 0$, find β and **H**.

10.19 Which of the following media may be treated as conducting at 8 MHz?

(a) Wet marshy soil ($\varepsilon = 15\varepsilon_o$, $\mu = \mu_o$, $\sigma = 10^{-2}$ S/m)
(b) Intrinsic germanium ($\varepsilon = 16\varepsilon_o$, $\mu = \mu_o$, $\sigma = 0.025$ S/m)
(c) Sea water ($\varepsilon = 81\varepsilon_o$, $\mu = \mu_o$, $\sigma = 25$ S/m)

10.20 Calculate the skin depth and the velocity of propagation for a uniform plane wave at frequency 6 MHz traveling in polyvinylchloride ($\mu_r = 1$, $\varepsilon_r = 4$, $\tan \theta_\eta = 7 \times 10^{-2}$).

10.21 A uniform plane wave in a lossy medium has a phase constant of 1.6 rad/m at 10^7 Hz and its magnitude is reduced by 60% for every 2 m traveled. Find the skin depth and speed of the wave.

10.22 (a) Determine the dc resistance of a round copper wire ($\sigma = 5.8 \times 10^7$ S/m, $\mu_r = 1$, $\varepsilon_r = 1$) of radius 1.2 mm and length 600 m.
(b) Find the ac resistance at 100 MHz.
(c) Calculate the approximate frequency where dc and ac resistances are equal.

10.23 A 40-m-long aluminum ($\sigma = 3.5 \times 10^7$ S/m, $\mu_r = 1$, $\varepsilon_r = 1$) pipe with inner and outer radii 9 mm and 12 mm carries a total current of $6 \sin 10^6 \pi t$ A. Find the skin depth and the effective resistance of the pipe.

10.24 Show that in a good conductor, the skin depth δ is always much shorter than the wavelength.

10.25 Brass waveguides are often silver plated to reduce losses. If at least the thickness of silver ($\mu = \mu_o$, $\varepsilon = \varepsilon_o$, $\sigma = 6.1 \times 10^7$ S/m) must be 5δ, find the minimum thickness required for a waveguide operating at 12 GHz.

10.26 A uniform plane wave in a lossy nonmagnetic media has

$$\mathbf{E}_s = (5\mathbf{a}_x + 12\mathbf{a}_y)\mathbf{e}^{-\gamma z}, \ \gamma = 0.2 + j3.4/\text{m}$$

(a) Compute the magnitude of the wave at $z = 4$ m.

(b) Find the loss in dB suffered by the wave in the interval $0 < z < 3$ m.

(c) Calculate the Poynting vector at $z = 4$, $t = T/8$. Take $\omega = 10^8$ rad/s.

10.27 In a nonmagnetic material,

$$\mathbf{H} = 30 \cos (2\pi \times 10^8 t - 6x)\, \mathbf{a}_y \text{ mA/m}$$

find: (a) the intrinsic impedance, (b) the Poynting vector, (c) the time-average power crossing the surface $x = 1, 0 < y < 2, 0 < z < 3$ m.

***10.28** Show that eqs. (10.67) and (10.68) are equivalent.

10.29 In a transmission line filled with a lossless dielectric ($\varepsilon = 4.5\varepsilon_o$, $\mu = \mu_o$),

$$\mathbf{E} = \frac{40}{\rho} \sin (\omega t - 2z)\, \mathbf{a}_\rho \text{ V/m}$$

find: (a) ω and $\mathbf{H}$, (b) the Poynting vector, (c) the total time-average power crossing the surface $z = 1$ m, 2 mm $< \rho <$ 3 mm, $0 < \phi < 2\pi$.

10.30 (a) For a normal incidence upon the dielectric–dielectric interface for which $\mu_1 = \mu_2 = \mu_o$, we define R and T as the reflection and transmission coefficients for average powers, i.e., $P_{r,\text{ave}} = RP_{i,\text{ave}}$ and $P_{t,\text{ave}} = TP_{i,\text{ave}}$. Prove that

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \quad \text{and} \quad T = \frac{4n_1n_2}{(n_1 + n_2)^2}$$

where n_1 and n_2 are the reflective indices of the media.

(b) Determine the ratio n_1/n_2 so that the reflected and the transmitted waves have the same average power.

10.31 The plane wave $\mathbf{E} = 30 \cos(\omega t - z)\mathbf{a}_x$ V/m in air normally hits a lossless medium ($\mu = \mu_o$, $\varepsilon = 4\varepsilon_o$) at $z = 0$. (a) Find Γ, τ, and s. (b) Calculate the reflected electric and magnetic fields.

10.32 A uniform plane wave in air with

$$\mathbf{H} = 4 \sin (\omega t - 5x)\, \mathbf{a}_y \text{ A/m}$$

is normally incident on a plastic region with the parameters $\mu = \mu_o$, $\varepsilon = 4\varepsilon_o$, and $\sigma = 0$. (a) Obtain the total electric field in air. (b) Calculate the time-average power density in the plastic region. (c) Find the standing wave ratio.

10.33 A plane wave in free space with $\mathbf{E} = 3.6 \cos(\omega t - 3x)\,\mathbf{a}_y$ V/m is incident normally on an interface at $x = 0$. If a lossless medium with $\sigma = 0$, $\varepsilon_r = 12.5$ exits for $x \geq 0$ and the reflected wave has $\mathbf{H}_r = -1.2 \cos(\omega t + 3x)\,\mathbf{a}_z$ mA/m, find μ_2.

10.34 Region 1 is a lossless medium for which $y \geq 0$, $\mu = \mu_o$, $\varepsilon = 4\varepsilon_o$, whereas region 2 is free space, $y \leq 0$. If a plane wave $\mathbf{E} = 5 \cos(10^8 t + \beta y)\,\mathbf{a}_z$ V/m exists in region 1, find: (a) the total electric field component of the wave in region 2, (b) the time-average Poynting vector in region 1, (c) the time-average Poynting vector in region 2.

10.35 A plane wave in free space ($z \leq 0$) is incident normally on a large block of material with $\varepsilon_r = 12$, $\mu_r = 3$, $\sigma = 0$ which occupies $z \geq 0$. If the incident electric field is

$$\mathbf{E} = 30 \cos(\omega t - z)\,\mathbf{a}_y \text{ V/m}$$

find: (a) ω, (b) the standing wave ratio, (c) the reflected magnetic field, (d) the average power density of the transmitted wave.

10.36 A 30-MHz uniform plane wave with

$$\mathbf{H} = 10 \sin(\omega t + \beta x)\,\mathbf{a}_z \text{ mA/m}$$

exists in region $x \geq 0$ having $\sigma = 0$, $\varepsilon = 9\varepsilon_o$, $\mu = 4\mu_o$. At $x = 0$, the wave encounters free space. Determine (a) the polarization of the wave, (b) the phase constant β, (c) the displacement current density in region $x \geq 0$, (d) the reflected and transmitted magnetic fields, and (e) the average power density in each region.

10.37 A uniform plane wave in air is normally incident on an infinite lossless dielectric material having $\varepsilon = 3\varepsilon_o$ and $\mu = \mu_o$. If the incident wave is $\mathbf{E}_i = 10 \cos(\omega t - z)\,\mathbf{a}_y$ V/m, find:

(a) λ and ω of the wave in air and the transmitted wave in the dielectric medium
(b) The incident $\mathbf{H}_i$ field
(c) Γ and τ
(d) The total electric field and the time-average power in both regions

***10.38** A signal in air ($z \geq 0$) with the electric field component

$$\mathbf{E} = 10 \sin(\omega t + 3z)\,\mathbf{a}_x \text{ V/m}$$

hits normally the ocean surface at $z = 0$ as in Figure 10.19. Assuming that the ocean surface is smooth and that $\varepsilon = 80\varepsilon_o$, $\mu = \mu_o$, $\sigma = 4$ mhos/m in ocean, determine

(a) ω
(b) The wavelength of the signal in air
(c) The loss tangent and intrinsic impedance of the ocean
(d) The reflected and transmitted $\mathbf{E}$ field

10.39 Sketch the standing wave in eq. (10.87) at $t = 0$, $T/8$, $T/4$, $3T/8$, $T/2$, and so on, where $T = 2\pi/\omega$.

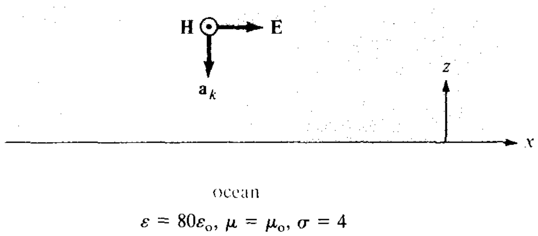

Figure 10.19 For Problem 10.38.

10.40 A uniform plane wave is incident at an angle $\theta_i = 45°$ on a pair of dielectric slabs joined together as shown in Figure 10.20. Determine the angles of transmission θ_{t1} and θ_{t2} in the slabs.

10.41 Show that the field

$$\mathbf{E}_s = 20 \sin (k_x x) \cos (k_y y) \, \mathbf{a}_z$$

where $k_x^2 + k_y^2 = \omega^2 \mu_o \varepsilon_o$, can be represented as the superposition of four propagating plane waves. Find the corresponding $\mathbf{H}_s$.

10.42 Show that for nonmagnetic dielectric media, the reflection and transmission coefficients for oblique incidence become

$$\Gamma_{\|} = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)}, \qquad \tau_{\|} = \frac{2 \cos\theta_i \sin\theta_t}{\sin(\theta_t + \theta_i)\cos(\theta_t - \theta_i)}$$

$$\Gamma_{\perp} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}, \qquad \tau_{\perp} = \frac{2 \cos\theta_i \sin\theta_t}{\sin(\theta_t + \theta_i)}$$

***10.43** A parallel-polarized wave in air with

$$\mathbf{E} = (8\mathbf{a}_y - 6\mathbf{a}_z) \sin(\omega t - 4y - 3z) \text{ V/m}$$

impinges a dielectric half-space as shown in Figure 10.21. Find: (a) the incidence angle θ_i, (b) the time average in air ($\mu = \mu_o$, $\varepsilon = \varepsilon_o$), (c) the reflected and transmitted **E** fields.

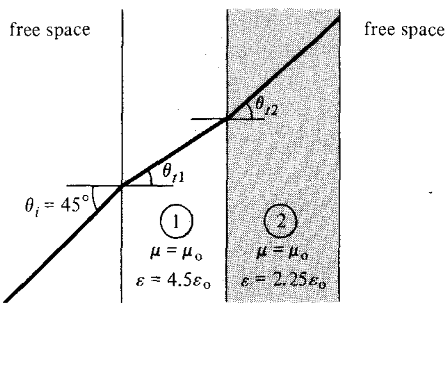

Figure 10.20 For Problem 10.40.

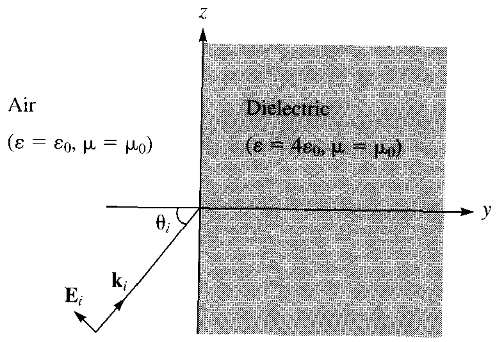

Figure 10.21 For Problem 10.43.

10.44 In a dielectric medium ($\varepsilon = 9\varepsilon_o$, $\mu = \mu_o$), a plane wave with

$$\mathbf{H} = 0.2 \cos (10^9 t - kx - k\sqrt{8}z)\, \mathbf{a}_y \text{ A/m}$$

is incident on an air boundary at $z = 0$, find

(a) θ_r and θ_t

(b) k

(c) The wavelength in the dielectric and air

(d) The incident **E**

(e) The transmitted and reflected **E**

(f) The Brewster angle

***10.45** A plane wave in air with

$$\mathbf{E} = (8\mathbf{a}_x + 6\mathbf{a}_y + 5\mathbf{a}_z) \sin (\omega t + 3x - 4y) \text{ V/m}$$

is incident on a copper slab in $y \geq 0$. Find ω and the reflected wave. Assume copper is a perfect conductor. (*Hint:* Write down the field components in both media and match the boundary conditions.)

10.46 A polarized wave is incident from air to polystyrene with $\mu = \mu_o$, $\varepsilon = 2.6\varepsilon$ at Brewster angle. Determine the transmission angle.

Chapter 11

TRANSMISSION LINES

There is a story about four men named Everybody, Somebody, Anybody, and Nobody. There was an important job to be done, and Everybody was asked to do it. Everybody was sure that Somebody would do it. Anybody could have done it, but Nobody did it. Somebody got angry about that, because it was Everybody's job. Everybody thought that Anybody could do it, and Nobody realized that Everybody wouldn't do it. It ended up that Everybody blamed Somebody, when actually Nobody did what Anybody could have done.

—ANONYMOUS

11.1 INTRODUCTION

Our discussion in the previous chapter was essentially on wave propagation in unbounded media, media of infinite extent. Such wave propagation is said to be unguided in that the uniform plane wave exists throughout all space and EM energy associated with the wave spreads over a wide area. Wave propagation in unbounded media is used in radio or TV broadcasting, where the information being transmitted is meant for everyone who may be interested. Such means of wave propagation will not help in a situation like telephone conversation, where the information is received privately by one person.

Another means of transmitting power or information is by guided structures. Guided structures serve to guide (or direct) the propagation of energy from the source to the load. Typical examples of such structures are transmission lines and waveguides. Waveguides are discussed in the next chapter; transmission lines are considered in this chapter.

Transmission lines are commonly used in power distribution (at low frequencies) and in communications (at high frequencies). Various kinds of transmission lines such as the twisted-pair and coaxial cables (thinnet and thicknet) are used in computer networks such as the Ethernet and internet.

A transmission line basically consists of two or more parallel conductors used to connect a source to a load. The source may be a hydroelectric generator, a transmitter, or an oscillator; the load may be a factory, an antenna, or an oscilloscope, respectively. Typical transmission lines include coaxial cable, a two-wire line, a parallel-plate or planar line, a wire above the conducting plane, and a microstrip line. These lines are portrayed in Figure 11.1. Notice that each of these lines consists of two conductors in parallel. Coaxial cables are routinely used in electrical laboratories and in connecting TV sets to TV antennas. Microstrip lines (similar to that in Figure 11.1e) are particularly important in integrated circuits where metallic strips connecting electronic elements are deposited on dielectric substrates.

Transmission line problems are usually solved using EM field theory and electric circuit theory, the two major theories on which electrical engineering is based. In this

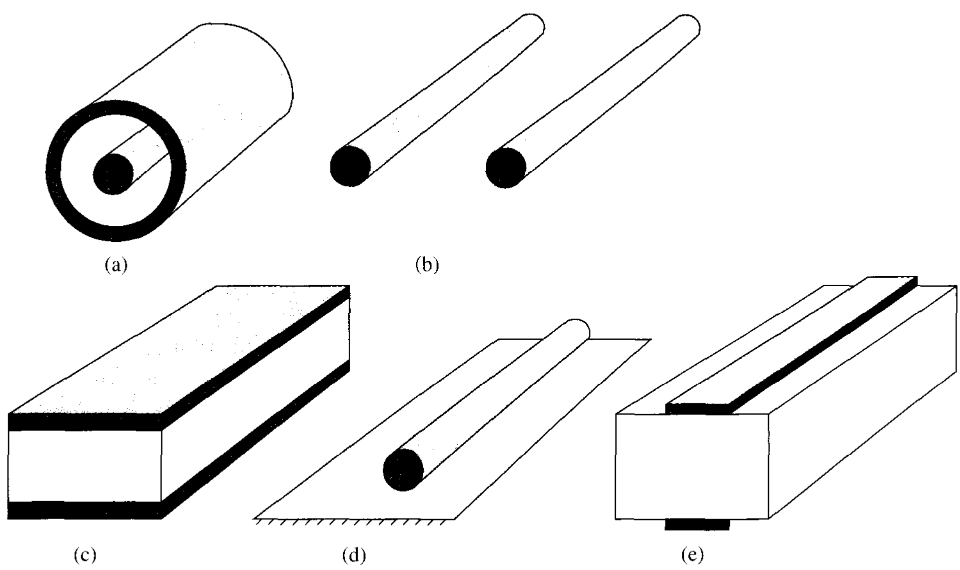

Figure 11.1 Cross-sectional view of typical transmission lines: **(a)** coaxial line, **(b)** two-wire line, **(c)** planar line, **(d)** wire above conducting plane, **(e)** microstrip line.

chapter, we use circuit theory because it is easier to deal with mathematically. The basic concepts of wave propagation (such as propagation constant, reflection coefficient, and standing wave ratio) covered in the previous chapter apply here.

Our analysis of transmission lines will include the derivation of the transmission-line equations and characteristic quantities, the use of the Smith chart, various practical applications of transmission lines, and transients on transmission lines.

11.2 TRANSMISSION LINE PARAMETERS

It is customary and convenient to describe a transmission line in terms of its line parameters, which are its resistance per unit length R, inductance per unit length L, conductance per unit length G, and capacitance per unit length C. Each of the lines shown in Figure 11.1 has specific formulas for finding R, L, G, and C. For coaxial, two-wire, and planar lines, the formulas for calculating the values of R, L, G, and C are provided in Table 11.1 The dimensions of the lines are as shown in Figure 11.2. Some of the formulas[1] in Table 11.1 were derived in Chapters 6 and 8. It should be noted that

1. The line parameters R, L, G, and C are not discrete or lumped but distributed as shown in Figure 11.3. By this we mean that the parameters are uniformly distributed along the entire length of the line.

[1]Similar formulas for other transmission lines can be obtained from engineering handbooks or data books—e.g., M. A. R. Guston, *Microwave Transmission-line Impedance Data.* London: Van Nostrand Reinhold, 1972.

TABLE 11.1 Distributed Line Parameters at High Frequencies*

Parameters	Coaxial Line	Two-Wire Line	Planar Line
R (Ω/m)	$\frac{1}{2\pi\delta\sigma_c}\left[\frac{1}{a}+\frac{1}{b}\right]$ $(\delta \ll a, c - b)$	$\frac{1}{\pi a\delta\sigma_c}$ $(\delta \ll a)$	$\frac{2}{w\delta\sigma_c}$ $(\delta \ll t)$
L (H/m)	$\frac{\mu}{2\pi}\ln\frac{b}{a}$	$\frac{\mu}{\pi}\cosh^{-1}\frac{d}{2a}$	$\frac{\mu d}{w}$
G (S/m)	$\frac{2\pi\sigma}{\ln\frac{b}{a}}$	$\frac{\pi\sigma}{\cosh^{-1}\frac{d}{2a}}$	$\frac{\sigma w}{d}$
C (F/m)	$\frac{2\pi\varepsilon}{\ln\frac{b}{a}}$	$\frac{\pi\varepsilon}{\cosh^{-1}\frac{d}{2a}}$	$\frac{\varepsilon w}{d}$ $(w \gg d)$

$*\delta = \frac{1}{\sqrt{\pi f \mu_c \sigma_c}}$ = skin depth of the conductor; $\cosh^{-1}\frac{d}{2a} \simeq \ln\frac{d}{a}$ if $\left[\frac{d}{2a}\right]^2 \gg 1$.

2. For each line, the conductors are characterized by σ_c, μ_c, $\varepsilon_c = \varepsilon_o$, and the homogeneous dielectric separating the conductors is characterized by σ, μ, ε.
3. $G \neq 1/R$; R is the ac resistance per unit length of the conductors comprising the line and G is the conductance per unit length due to the dielectric medium separating the conductors.
4. The value of L shown in Table 11.1 is the external inductance per unit length; that is, $L = L_{\text{ext}}$. The effects of internal inductance L_{in} $(= R/\omega)$ are negligible as high frequencies at which most communication systems operate.
5. For each line,

$$LC = \mu\varepsilon \qquad \text{and} \qquad \frac{G}{C} = \frac{\sigma}{\varepsilon} \tag{11.1}$$

As a way of preparing for the next section, let us consider how an EM wave propagates through a two-conductor transmission line. For example, consider the coaxial line connecting the generator or source to the load as in Figure 11.4(a). When switch S is closed,

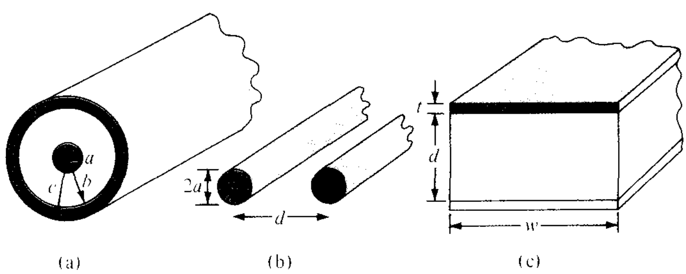

Figure 11.2 Common transmission lines: (a) coaxial line, (b) two-wire line, (c) planar line.

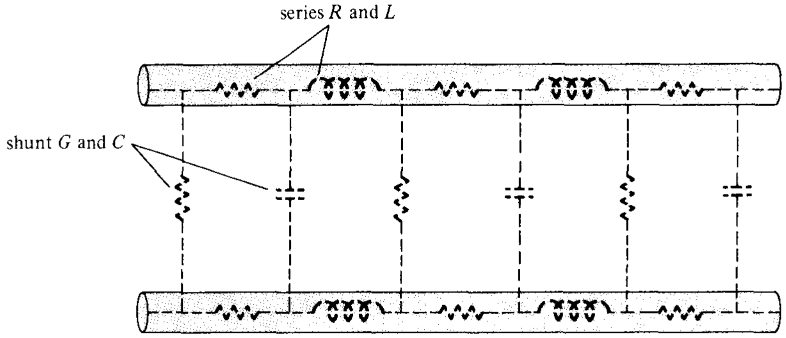

Figure 11.3 Distributed parameters of a two-conductor transmission line.

the inner conductor is made positive with respect to the outer one so that the **E** field is radially outward as in Figure 11.4(b). According to Ampere's law, the **H** field encircles the current carrying conductor as in Figure 11.4(b). The Poynting vector (**E** × **H**) points along the transmission line. Thus, closing the switch simply establishes a disturbance, which appears as a transverse electromagnetic (TEM) wave propagating along the line. This wave is a nonuniform plane wave and by means of it power is transmitted through the line.

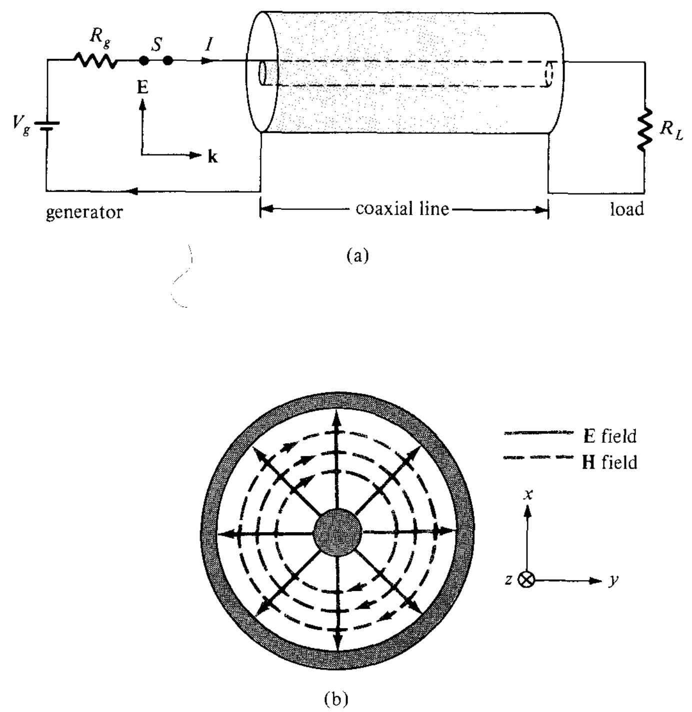

Figure 11.4 **(a)** Coaxial line connecting the generator to the load; **(b)** **E** and **H** fields on the coaxial line.

11.3 TRANSMISSION LINE EQUATIONS

As mentioned in the previous section, a two-conductor transmission line supports a TEM wave; that is, the electric and magnetic fields on the line are transverse to the direction of wave propagation. An important property of TEM waves is that the fields **E** and **H** are uniquely related to voltage V and current I, respectively:

$$V = -\int \mathbf{E} \cdot d\mathbf{l}, \qquad I = \oint \mathbf{H} \cdot d\mathbf{l} \tag{11.2}$$

In view of this, we will use circuit quantities V and I in solving the transmission line problem instead of solving field quantities **E** and **H** (i.e., solving Maxwell's equations and boundary conditions). The circuit model is simpler and more convenient.

Let us examine an incremental portion of length Δz of a two-conductor transmission line. We intend to find an equivalent circuit for this line and derive the line equations. From Figure 11.3, we expect the equivalent circuit of a portion of the line to be as in Figure 11.5. The model in Figure 11.5 is in terms of the line parameters R, L, G, and C, and may represent any of the two-conductor lines of Figure 11.3. The model is called the L-type equivalent circuit; there are other possible types (see Problem 11.1). In the model of Figure 11.5, we assume that the wave propagates along the $+z$-direction, from the generator to the load.

By applying Kirchhoff's voltage law to the outer loop of the circuit in Figure 11.5, we obtain

$$V(z, t) = R\,\Delta z\, I(z, t) + L\,\Delta z \frac{\partial I(z, t)}{\partial t} + V(z + \Delta z, t)$$

or

$$\frac{V(z + \Delta z, t) - V(z, t)}{\Delta z} = R\, I(z, t) + L \frac{\partial I(z, t)}{\partial t} \tag{11.3}$$

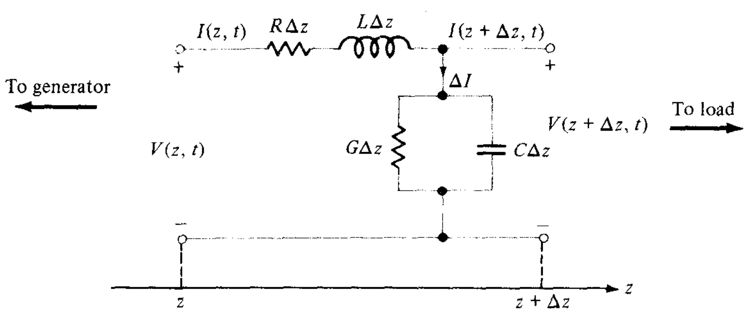

Figure 11.5 L-type equivalent circuit model of a differential length Δz of a two-conductor transmission line.

Taking the limit of eq. (11.3) as $\Delta z \to 0$ leads to

$$-\frac{\partial V(z, t)}{\partial z} = RI(z, t) + L\frac{\partial I(z, t)}{\partial t} \tag{11.4}$$

Similarly, applying Kirchoff's current law to the main node of the circuit in Figure 11.5 gives

$$\begin{aligned} I(z, t) &= I(z + \Delta z, t) + \Delta I \\ &= I(z + \Delta z, t) + G\,\Delta z\, V(z + \Delta z, t) + C\,\Delta z\frac{\partial V(z + \Delta z, t)}{\partial t} \end{aligned}$$

or

$$-\frac{I(z + \Delta z, t) - I(z, t)}{\Delta z} = G\,V(z + \Delta z, t) + C\frac{\partial V(z + \Delta z, t)}{\partial t} \tag{11.5}$$

As $\Delta z \to 0$, eq. (11.5) becomes

$$-\frac{\partial I(z, t)}{\partial z} = G\,V(z, t) + C\frac{\partial V(z, t)}{\partial t} \tag{11.6}$$

If we assume harmonic time dependence so that

$$V(z, t) = \text{Re}\,[V_s(z)\, e^{j\omega t}] \tag{11.7a}$$

$$I(z, t) = \text{Re}\,[I_s(z)\, e^{j\omega t}] \tag{11.7b}$$

where $V_s(z)$ and $I_s(z)$ are the phasor forms of $V(z, t)$ and $I(z, t)$, respectively, eqs. (11.4) and (11.6) become

$$-\frac{dV_s}{dz} = (R + j\omega L)\, I_s \tag{11.8}$$

$$-\frac{dI_s}{dz} = (G + j\omega C)\, V_s \tag{11.9}$$

In the differential eqs. (11.8) and (11.9), V_s and I_s are coupled. To separate them, we take the second derivative of V_s in eq. (11.8) and employ eq. (11.9) so that we obtain

$$\frac{d^2V_s}{dz^2} = (R + j\omega L)(G + j\omega C)\, V_s$$

or

$$\frac{d^2V_s}{dz^2} - \gamma^2 V_s = 0 \tag{11.10}$$

where

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \tag{11.11}$$

By taking the second derivative of I_s in eq. (11.9) and employing eq. (11.8), we get

$$\frac{d^2 I_s}{dz^2} - \gamma^2 I_s = 0 \tag{11.12}$$

We notice that eqs. (11.10) and (11.12) are, respectively, the wave equations for voltage and current similar in form to the wave equations obtained for plane waves in eqs. (10.17) and (10.19). Thus, in our usual notations, γ in eq. (11.11) is the propagation constant (in per meter), α is the attenuation constant (in nepers per meter or decibels[2] per meter), and β is the phase constant (in radians per meter). The wavelength λ and wave velocity u are, respectively, given by

$$\lambda = \frac{2\pi}{\beta} \tag{11.13}$$

$$u = \frac{\omega}{\beta} = f\lambda \tag{11.14}$$

The solutions of the linear homogeneous differential equations (11.10) and (11.12) are similar to Case 2 of Example 6.5, namely,

$$V_s(z) = \underset{\longrightarrow +z}{V_o^+ e^{-\gamma z}} + \underset{-z \longleftarrow}{V_o^- e^{\gamma z}} \tag{11.15}$$

and

$$I_s(z) = \underset{\longrightarrow +z}{I_o^+ e^{-\gamma z}} + \underset{-z \longleftarrow}{I_o^- e^{\gamma z}} \tag{11.16}$$

where V_o^+, V_o^-, I_o^+, and I_o^- are wave amplitudes; the + and − signs, respectively, denote wave traveling along $+z$- and $-z$-directions, as is also indicated by the arrows. Thus, we obtain the instantaneous expression for voltage as

$$\begin{aligned} V(z, t) &= \text{Re}\,[V_s(z)\, e^{j\omega t}] \\ &= V_o^+ e^{-\alpha z} \cos(\omega t - \beta z) + V_o^- e^{\alpha z} \cos(\omega t + \beta z) \end{aligned} \tag{11.17}$$

The **characteristic impedance** Z_o of the line is the ratio of positively traveling voltage wave to current wave at any point on the line.

---

[2]Recall from eq. (10.35) that 1 Np = 8.686 dB.

Z_o is analogous to η, the intrinsic impedance of the medium of wave propagation. By substituting eqs. (11.15) and (11.16) into eqs. (11.8) and (11.9) and equating coefficients of terms $e^{\gamma z}$ and $e^{-\gamma z}$, we obtain

$$Z_o = \frac{V_o^+}{I_o^+} = -\frac{V_o^-}{I_o^-} = \frac{R + j\omega L}{\gamma} = \frac{\gamma}{G + j\omega C} \tag{11.18}$$

or

$$\boxed{Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = R_o + jX_o} \tag{11.19}$$

where R_o and X_o are the real and imaginary parts of Z_o. R_o should not be mistaken for R—while R is in ohms per meter; R_o is in ohms. The propagation constant γ and the characteristic impedance Z_o are important properties of the line because they both depend on the line parameters R, L, G, and C and the frequency of operation. The reciprocal of Z_o is the characteristic admittance Y_o, that is, $Y_o = 1/Z_o$.

The transmission line considered thus far in this section is the *lossy* type in that the conductors comprising the line are imperfect ($\sigma_c \neq \infty$) and the dielectric in which the conductors are embedded is lossy ($\sigma \neq 0$). Having considered this general case, we may now consider two special cases of lossless transmission line and distortionless line.

A. Lossless Line ($R = 0 = G$)

A **transmission line** is said to be **lossless** if the conductors of the line are perfect ($\sigma_c \approx \infty$) and the dielectric medium separating them is lossless ($\sigma \simeq 0$).

For such a line, it is evident from Table 11.1 that when $\sigma_c \simeq \infty$ and $\sigma \simeq 0$.

$$\boxed{R = 0 = G} \tag{11.20}$$

This is a necessary condition for a line to be lossless. Thus for such a line, eq. (11.20) forces eqs. (11.11), (11.14), and (11.19) to become

$$\alpha = 0, \qquad \gamma = j\beta = j\omega\sqrt{LC} \tag{11.21a}$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = f\lambda \tag{11.21b}$$

$$X_o = 0, \qquad Z_o = R_o = \sqrt{\frac{L}{C}} \tag{11.21c}$$

B. Distortionless Line ($R/L = G/C$)

A signal normally consists of a band of frequencies; wave amplitudes of different frequency components will be attenuated differently in a lossy line as α is frequency dependent. This results in distortion.

A **distortionless line** is one in which the attenuation constant α is frequency independent while the phase constant β is linearly dependent on frequency.

From the general expression for α and β [in eq. (11.11)], a distortionless line results if the line parameters are such that

$$\boxed{\frac{R}{L} = \frac{G}{C}} \tag{11.22}$$

Thus, for a distortionless line,

$$\begin{aligned}\gamma &= \sqrt{RG\left(1 + \frac{j\omega L}{R}\right)\left(1 + \frac{j\omega C}{G}\right)} \\ &= \sqrt{RG}\left(1 + \frac{j\omega C}{G}\right) = \alpha + j\beta\end{aligned}$$

or

$$\alpha = \sqrt{RG}, \qquad \beta = \omega\sqrt{LC} \tag{11.23a}$$

showing that α does not depend on frequency whereas β is a linear function of frequency. Also

$$Z_o = \sqrt{\frac{R(1 + j\omega L/R)}{G(1 + j\omega C/G)}} = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}} = R_o + jX_o$$

or

$$R_o = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}, \qquad X_o = 0 \tag{11.23b}$$

and

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = f\lambda \tag{11.23c}$$

Note that

1. The phase velocity is independent of frequency because the phase constant β linearly depends on frequency. We have shape distortion of signals unless α and u are independent of frequency.
2. u and Z_o remain the same as for lossless lines.

TABLE 11.2 Transmission Line Characteristics

Case	Propagation Constant $\gamma = \alpha + j\beta$	Characteristic Impedance $Z_o = R_o + jX_o$
General	$\sqrt{(R + j\omega L)(G + j\omega C)}$	$\sqrt{\dfrac{R + j\omega L}{G + j\omega C}}$
Lossless	$0 + j\omega\sqrt{LC}$	$\sqrt{\dfrac{L}{C}} + j0$
Distortionless	$\sqrt{RG} + j\omega\sqrt{LC}$	$\sqrt{\dfrac{L}{C}} + j0$

3. A lossless line is also a distortionless line, but a distortionless line is not necessarily lossless. Although lossless lines are desirable in power transmission, telephone lines are required to be distortionless.

A summary of our discussion is in Table 11.2. For the greater part of our analysis, we shall restrict our discussion to lossless transmission lines.

EXAMPLE 11.1

An air line has characteristic impedance of 70 Ω and phase constant of 3 rad/m at 100 MHz. Calculate the inductance per meter and the capacitance per meter of the line.

Solution:

An air line can be regarded as a lossless line since $\sigma \simeq 0$. Hence

$$R = 0 = G \qquad \text{and} \qquad \alpha = 0$$

$$Z_o = R_o = \sqrt{\frac{L}{C}} \tag{11.1.1}$$

$$\beta = \omega \sqrt{LC} \tag{11.1.2}$$

Dividing eq. (11.1.1) by eq. (11.1.2) yields

$$\frac{R_o}{\beta} = \frac{1}{\omega C}$$

or

$$C = \frac{\beta}{\omega R_o} = \frac{3}{2\pi \times 100 \times 10^6(70)} = 68.2 \text{ pF/m}$$

From eq. (11.1.1),

$$L = R_o^2 C = (70)^2(68.2 \times 10^{-12}) = 334.2 \text{ nH/m}$$

PRACTICE EXERCISE 11.1

A transmission line operating at 500 MHz has $Z_o = 80\ \Omega$, $\alpha = 0.04$ Np/m, $\beta = 1.5$ rad/m. Find the line parameters R, L, G, and C.

Answer: 3.2 Ω/m, 38.2 nH/m, 5×10^{-4} S/m, 5.97 pF/m.

EXAMPLE 11.2

A distortionless line has $Z_o = 60\ \Omega$, $\alpha = 20$ mNp/m, $u = 0.6c$, where c is the speed of light in a vacuum. Find R, L, G, C, and λ at 100 MHz.

Solution:

For a distortionless line,

$$RC = GL \quad \text{or} \quad G = \frac{RC}{L}$$

and hence

$$Z_o = \sqrt{\frac{L}{C}} \tag{11.2.1}$$

$$\alpha = \sqrt{RG} = R\sqrt{\frac{C}{L}} = \frac{R}{Z_o} \tag{11.2.2a}$$

or

$$R = \alpha Z_o \tag{11.2.2b}$$

But

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \tag{11.2.3}$$

From eq. (11.2.2b),

$$R = \alpha Z_o = (20 \times 10^{-3})(60) = 1.2\ \Omega/\text{m}$$

Dividing eq. (11.2.1) by eq. (11.2.3) results in

$$L = \frac{Z_o}{u} = \frac{60}{0.6\,(3 \times 10^8)} = 333\ \text{nH/m}$$

From eq. (11.2.2a),

$$G = \frac{\alpha^2}{R} = \frac{400 \times 10^{-6}}{1.2} = 333\ \mu\text{S/m}$$

Multiplying eqs. (11.2.1) and (11.2.3) together gives

$$uZ_o = \frac{1}{C}$$

or

$$C = \frac{1}{uZ_o} = \frac{1}{0.6\,(3 \times 10^8)\,60} = 92.59 \text{ pF/m}$$

$$\lambda = \frac{u}{f} = \frac{0.6\,(3 \times 10^8)}{10^8} = 1.8 \text{ m}$$

PRACTICE EXERCISE 11.2

A telephone line has $R = 30\ \Omega$/km, $L = 100$ mH/km, $G = 0$, and $C = 20\ \mu$F/km. At $f = 1$ kHz, obtain:

(a) The characteristic impedance of the line

(b) The propagation constant

(c) The phase velocity

Answer: (a) $70.75\underline{/-1.367^\circ}\ \Omega$, (b) $2.121 \times 10^{-4} + j8.888 \times 10^{-3}$/m, (c) 7.069×105 m/s.

11.4 INPUT IMPEDANCE, SWR, AND POWER

Consider a transmission line of length ℓ, characterized by γ and Z_o, connected to a load Z_L as shown in Figure 11.6. Looking into the line, the generator sees the line with the load as an input impedance Z_{in}. It is our intention in this section to determine the input impedance, the standing wave ratio (SWR), and the power flow on the line.

Let the transmission line extend from $z = 0$ at the generator to $z = \ell$ at the load. First of all, we need the voltage and current waves in eqs. (11.15) and (11.16), that is

$$V_s(z) = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z} \tag{11.24}$$

$$I_s(z) = \frac{V_o^+}{Z_o} e^{-\gamma z} - \frac{V_o^-}{Z_o} e^{\gamma z} \tag{11.25}$$

where eq. (11.18) has been incorporated. To find V_o^+ and V_o^-, the terminal conditions must be given. For example, if we are given the conditions at the input, say

$$V_o = V(z = 0), \qquad I_o = I(z = 0) \tag{11.26}$$

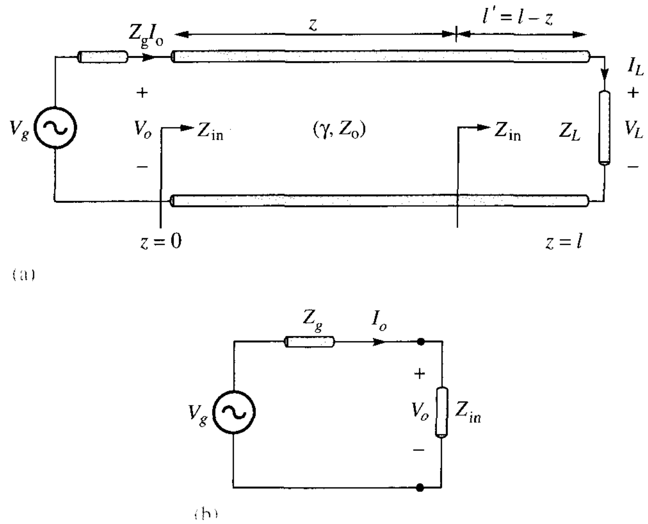

Figure 11.6 (a) Input impedance due to a line terminated by a load; **(b)** equivalent circuit for finding V_o and I_o in terms of Z_{in} at the input.

substituting these into eqs. (11.24) and (11.25) results in

$$V_o^+ = \frac{1}{2}(V_o + Z_o I_o) \tag{11.27a}$$

$$V_o^- = \frac{1}{2}(V_o - Z_o I_o) \tag{11.27b}$$

If the input impedance at the input terminals is Z_{in}, the input voltage V_o and the input current I_o are easily obtained from Figure 11.6(b) as

$$V_o = \frac{Z_{in}}{Z_{in} + Z_g} V_g, \qquad I_o = \frac{V_g}{Z_{in} + Z_g} \tag{11.28}$$

On the other hand, if we are given the conditions at the load, say

$$V_L = V(z = \ell), \qquad I_L = I(z = \ell) \tag{11.29}$$

Substituting these into eqs. (11.24) and (11.25) gives

$$V_o^+ = \frac{1}{2}(V_L + Z_o I_L)e^{\gamma\ell} \tag{11.30a}$$

$$V_o^- = \frac{1}{2}(V_L - Z_o I_L)e^{-\gamma\ell} \tag{11.30b}$$

Next, we determine the input impedance $Z_{in} = V_s(z)/I_s(z)$ at any point on the line. At the generator, for example, eqs. (11.24) and (11.25) yield

$$Z_{in} = \frac{V_s(z)}{I_s(z)} = \frac{Z_o(V_o^+ + V_o^-)}{V_o^+ - V_o^-} \tag{11.31}$$

Substituting eq. (11.30) into (11.31) and utilizing the fact that

$$\frac{e^{\gamma\ell} + e^{-\gamma\ell}}{2} = \cosh \gamma\ell, \qquad \frac{e^{\gamma\ell} - e^{-\gamma\ell}}{2} = \sinh \gamma\ell \tag{11.32a}$$

or

$$\tanh \gamma\ell = \frac{\sinh \gamma\ell}{\cosh \gamma\ell} = \frac{e^{\gamma\ell} - e^{-\gamma\ell}}{e^{\gamma\ell} + e^{-\gamma\ell}} \tag{11.32b}$$

we get

$$\boxed{Z_{in} = Z_o \left[\frac{Z_L + Z_o \tanh \gamma\ell}{Z_o + Z_L \tanh \gamma\ell}\right]} \qquad \text{(lossy)} \tag{11.33}$$

Although eq. (11.33) has been derived for the input impedance Z_{in} at the generation end, it is a general expression for finding Z_{in} at any point on the line. To find Z_{in} at a distance ℓ' from the load as in Figure 11.6(a), we replace ℓ by ℓ'. A formula for calculating the hyperbolic tangent of a complex number, required in eq. (11.33), is found in Appendix A.3.

For a lossless line, $\gamma = j\beta$, $\tanh j\beta\ell = j \tan \beta\ell$, and $Z_o = R_o$, so eq. (11.33) becomes

$$\boxed{Z_{in} = Z_o \left[\frac{Z_L + jZ_o \tan \beta\ell}{Z_o + jZ_L \tan \beta\ell}\right]} \qquad \text{(lossless)} \tag{11.34}$$

showing that the input impedance varies periodically with distance ℓ from the load. The quantity $\beta\ell$ in eq. (11.34) is usually referred to as the *electrical length* of the line and can be expressed in degrees or radians.

We now define Γ_L as the *voltage reflection coefficient* (at the load). Γ_L is the ratio of the voltage reflection wave to the incident wave at the load, that is,

$$\Gamma_L = \frac{V_o^- e^{\gamma\ell}}{V_o^+ e^{-\gamma\ell}} \tag{11.35}$$

Substituting V_o^- and V_o^+ in eq. (11.30) into eq. (11.35) and incorporating $V_L = Z_L I_L$ gives

$$\boxed{\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o}} \tag{11.36}$$